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FREE-EDGE AND CONTACT-EDGE SINGULARITIES IN LAMINATED COMPOSITES

April 1982

R. I. ZWIERS and T. C. T. TING
Department of Materials Engineering
University of Illinois at Chicago Circle
Chicago, Illinois 60680

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ABSTRACT

This effort is part of the predictive response capability development to provide design criteria for space vehicles. For example, mechanical joints in structure have stress risers around the bolt holes due to the singularities from linear analysis.

In this task the stress singularities at the free-edge and at the contact-edge of an interface in a laminated composite are studied. The composite is subjected to a uniform extension in the direction parallel to the plane of laminate layers. Two types of stress singularities are found. For the free-edge problem the stress solution for some fiber orientations contains only a singularity of $k^{*}r^{\delta}$, while the solution for other combinations of fiber orientation contains an additional singularity of $k(\ln r)$. For the contact-edge problem only $k^{*}r^{\delta}$ singularities occur for all fiber orientations.

FOREWORD

This research work was performed for the Army Materials and Mechanics Research Center (AMMRC), Watertown, Massachusetts under Contract No. DAAG 46-80-C-0081 with the University of Illinois at Chicago Circle, Chicago, Illinois. Mr. J. F. Dignam of the AMMRC was the project manager and Dr. S. C. Chou was the technical monitor. The support and encouragement of Mr. Dignam and Dr. Chou are gratefully acknowledged. The method outlined in section 4.1 was due to Dr. R. L. Spilker.

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Chapter I

INTRODUCTION

1.1 INTRODUCTION

A free-edge in a composite is the intersection of an interface plane (between any two layers) and the free surface of the composite, while a contact-edge is the intersection of an interface plane (between any two layers) and the contact surface of the composite. The unusually large and possibly infinte stress at the free-edge and the contact-edge is one of the factors responsible for delamination when the composite is subjected to external loading. Since more composites are now used in space vehicles, it is important to analyze the nature of the stress singularities at the free-edge and the contact-edge so that special attention can be paid to the design of mechanical fastenings and joints. The research reported here is part of the effort of the "predictive response capability development" program.

There have been many investigators who analyzed the stress near the free-edge [1-9]. An analytical solution which is valid for the whole composite is practically impossible to obtain. Several approximate numerical solutions are available which show a good agreement between each other for points away from the free-edge. For points near the free-edge, numerical solutions are not capable of predicting an infinite stress when it exists, and this is where the discrepancies between

various approximate solutions occur. Wang and Choi [8] used an eigenfunction expansion technique to determine the stress in the interface. However, the completeness of the eigenfunction expansion is an open question [10]. In fact, the existence of the logarithmic singularity discussed in this paper implies that the eigenfunction expansion in terms of r^{δ} powers may not be complete. It is doubtful that the addition of (ln r) terms would make the eigenfunction expansion complete. As pointed out in [10, 11], singular terms of $(\ln r)^2$ and $(\ln r)^3$ etc., may also exist.

For composites whose layers are isotropic elastic materials, use of the biharmonic function, or the Airy stress function, seems to be the universal approach in the analysis of the stress singularities. (See [10, 12, 13], for example). There appears to be no universal approach in analysing the stress singularities in anisotropic elastic materials. Lekhnitskii [14] introduced two stress functions to analyze general anisotropic materials. His approach was used by Wang and Choi [8] to study the thermal stresses at the interface in a layered composite. Green and Zerna [15] employed a complex function representation for the general solution. Their approach was used by Bogy [16] and Kuo and Bogy [17] in conjunction with a generalized Mellin transform to analyze stress singularities in an anisotropic wedge. In this paper we use the approach which was originated by Stroh [18] and further developed by Barnett and others [19-21] for studying the surface waves in anisotropic elastic materials.

While the nature of the singularity, be it $k^* r^\delta$ or $k(\ln r)$, is independent of the stacking sequence of the layers in the composite and the complete boundary conditions, the unknown constant k^* in the singular solution is not. This suggests that one might use a special finite element at the free-edge or at the contact-edge (with regular finite elements elsewhere) so that the exact nature of the singularity is prescribed in the special element while the unknown constants associated with this special element are determined by solving the complete boundary value problem. If k^* so obtained happens to be zero at a particular free-edge or contact-edge, there is no r^δ singularity at that point. The purpose of this paper is to provide the exact nature of the singularity at the free-edge and at the contact-edge.

In the following sections of Chapter I, the notation for the basic equations (strain-displacement, stress-strain and equilibrium equations) is explained, and expressions for the elasticity constants and elastic compliances of materials orthotropic in the $(\hat{x}_1, x_2, \hat{x}_3)$ coordinate system are given.

In Chapter II a free-edge problem which undergoes a two-dimensional plane strain deformation is analyzed. The assumed expressions for stress and displacement when applied to the boundary and continuity conditions result in a system of linear homogeneous equations. The eigenvalues δ of this system, both real and complex are listed, and the procedure used to find the complex values of δ is explained. The existence of a singularity of $k^* r^\delta$ for negative δ is discussed.

In Chapters III and IV the free-edge problem is analyzed for three-dimensional deformation. In Chapter III it is established that the superposition of the uniform extension term ε_3 on the two-dimensional problem of Chapter II is a first order approximation to a three-dimensional deformation. In Chapter IV it is shown that the addition of the uniform extension term to the equations of Chapter II results in a system of non-homogeneous linear equations. Two methods are then used to solve for the assumed homogeneous stress. These solutions are valid for ($\theta/-\theta$) and (0/90) composites. A modified method is then introduced to solve for the non-homogeneous stress of the other (θ/θ') combinations. The existence of the $k(\ln r)$ singularity is discussed, and the logarithmic stress intensity factor k is listed for various combinations of (θ/θ'). Values of σ_{ij} which depend on the angle ϕ in polar coordinates are also listed.

In Chapter V a contact problem is analyzed. The altered boundary conditions lead to a new system of linear homogeneous equations for the two-dimensional plane strain problem. The roots δ, both real and complex, are listed and the existence of the singularity $k^* r^\delta$ is discussed. The three-dimensional deformation problem is then analyzed, and a particular solution which is uniform in stress is found and listed for some (θ/θ') combinations.

1.2 BASIC EQUATIONS

In a fixed rectangular coordinate system x_i, ($i = 1, 2, 3$), let u_i, σ_{ij} and ε_{ij} be the displacement, stress, and strain, respectively.

The strain-displacement, stress-strain and equilibrium equations may be written as

$$\varepsilon_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2 \tag{1.1}$$

$$\sigma_{ij} = c_{ijk\ell}\varepsilon_{k\ell} \tag{1.2}$$

or

$$\varepsilon_{ij} = s_{ijk\ell}\sigma_{k\ell} \tag{1.3}$$

$$\partial\sigma_{ij}/\partial x_j = 0 \tag{1.4}$$

where repeated indices imply summation, and

$$c_{ijk\ell} = c_{k\ell ij} = c_{jik\ell} \tag{1.5}$$

$$s_{ijk\ell} = s_{k\ell ij} = s_{jik\ell} \tag{1.6}$$

are the elasticity constants and the elastic compliances, respectively.

Due to the symmetric property of Eqs. (1.5, 1.6), one may rewrite Eqs. (1.2, 1.3, 1.5, 1.6) as

$$\sigma_q = c_{qt}\varepsilon_t, \qquad c_{qt} = c_{tq} \tag{1.7}$$

$$\varepsilon_q = s_{qt}\sigma_t, \qquad s_{qt} = s_{tq} \tag{1.8}$$

where

$$\left.\begin{array}{lll} \sigma_1 = \sigma_{11}, & \sigma_2 = \sigma_{22}, & \sigma_3 = \sigma_{33} \\ \sigma_4 = \sigma_{23}, & \sigma_5 = \sigma_{13}, & \sigma_6 = \sigma_{12} \end{array}\right\} \tag{1.9}$$

$$\left.\begin{array}{lll} \varepsilon_1 = \varepsilon_{11}, & \varepsilon_2 = \varepsilon_{22}, & \varepsilon_3 = \varepsilon_{33} \\ \varepsilon_4 = 2\varepsilon_{23}, & \varepsilon_5 = 2\varepsilon_{13}, & \varepsilon_6 = 2\varepsilon_{12} \end{array}\right\} \tag{1.10}$$

The transformation between $c_{ijk\ell}$ and c_{qt} are as follows

$$ij \text{ or } k\ell = \begin{cases} 11 \\ 22 \\ 33 \\ 23 \\ 31 \\ 12 \end{cases}, \qquad q \text{ or } t = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{cases} \tag{1.11}$$

Since the transformation between σ_{ij} and σ_i *is not identical* to that between ε_{ij} and ε_i, in order to represent the transformation between the subscripts of $s_{ijk\ell}$ and s_{qt}, Eq. (1.11) must be modified as follows. If either q or t is larger than 3, $s_{qt} = 2s_{ijk\ell}$. If both q and t are larger than 3, $s_{qt} = 4s_{ijk\ell}$; [22] contains further discussion about these transformations.

1.3 ORTHOTROPIC MATERIALS

For orthotropic materials with the (x_1, x_2, x_3) axes of symmetry, s_{ij} are zero except [25, 26]

$$\left.\begin{array}{l} s_{11} = 1/E_1, \quad s_{22} = 1/E_2, \quad s_{33} = 1/E_3, \\ s_{44} = 1/G_{23}, \quad s_{55} = 1/G_{31}, \quad s_{66} = 1/G_{12} \\ s_{12} = s_{21} = -\nu_{21}/E_2 \\ s_{13} = s_{31} = -\nu_{31}/E_3 \\ s_{23} = s_{32} = -\nu_{32}/E_3 \end{array}\right\} \tag{1.12}$$

where E_1, E_2, E_3, G_{23}, G_{31}, G_{12}, ν_{21}, ν_{31}, and ν_{32} are the engineering constants. The relation between s_{ij} and c_{ij} may be found in [22, 25, 26].

Consider a laminated composite which consists of a finite number of anisotropic elastic layers perfectly bonded at the interface, Fig. 1. Each layer of the composite lies in a plane parallel to the (x_1, x_3) plane. A cross section of the composite is shown in Fig. 2. This angle-ply graphite/epoxy laminated composite is assumed to be orthotropic with respect to the $(\hat{x}_1, x_2, \hat{x}_3)$ coordinate system. The $\hat{x}_3$-axis, which is the direction of the fibers (as shown in Fig. 3), makes an angle θ with the x_3-axis. It will be assumed that materials 1 and 2 in Fig. 2 are made of the same orthotropic material, but that the orientation of the fibers is different; the angles θ and θ' vary between layers. Therefore, $\hat{c}_{ij}$ and $\hat{s}_{ij}$ are the same for both materials, while c_{ij}, s_{ij} may not be. The relation between c_{ij} and $\hat{c}_{ij}$ may be found in [22]. The relations between s_{ij} and $\hat{s}_{ij}$ may be obtained from the relations between c_{ij} and $\hat{c}_{ij}$ by replacing c_{ij} by s_{ij} with the following exceptions. If either i or j is larger than 3, replace $2c_{ij}$ by s_{ij}. If both i and j are larger than 3, replace $4c_{ij}$ by s_{ij}. The same rules apply in replacing $\hat{c}_{ij}$ by $\hat{s}_{ij}$. Therefore we have

$$\left.\begin{aligned}
s_{11} &= c^4\hat{s}_{11} + c^2s^2(2\hat{s}_{13} + \hat{s}_{55}) + s^4\hat{s}_{33} \\
s_{12} &= c^2\hat{s}_{12} + s^2\hat{s}_{32} \\
s_{13} &= (c^4 + s^4)\hat{s}_{13} + c^2s^2(\hat{s}_{11} + \hat{s}_{33} - \hat{s}_{55}) \\
s_{15} &= cs[c^2(2\hat{s}_{13} + \hat{s}_{55} - 2\hat{s}_{11}) - s^2(2\hat{s}_{31} + \hat{s}_{55} - 2\hat{s}_{33})] \\
s_{22} &= \hat{s}_{22} \\
s_{23} &= c^2\hat{s}_{23} + s^2\hat{s}_{21} \\
s_{25} &= 2cs(\hat{s}_{23} - \hat{s}_{21}) \\
s_{33} &= c^4\hat{s}_{33} + c^2s^2(2\hat{s}_{13} + \hat{s}_{55}) + s^4\hat{s}_{11} \\
s_{35} &= cs[-c^2(2\hat{s}_{31} + \hat{s}_{55} - 2\hat{s}_{33}) + s^2(2\hat{s}_{13} + \hat{s}_{55} - 2\hat{s}_{11})] \\
s_{44} &= c^2\hat{s}_{44} + s^2\hat{s}_{66} \\
s_{46} &= cs(\hat{s}_{44} - \hat{s}_{66}) \\
s_{55} &= (c^2 - s^2)^2\hat{s}_{55} + 4c^2s^2(\hat{s}_{11} - 2\hat{s}_{13} + \hat{s}_{33}) \\
s_{66} &= c^2\hat{s}_{66} + s^2\hat{s}_{44}
\end{aligned}\right\} \quad (1.13)$$

where, for simplicity, the notation $c = \cos\theta$, $s = \sin\theta$ has been used. All other terms of s_{ij} are zero.

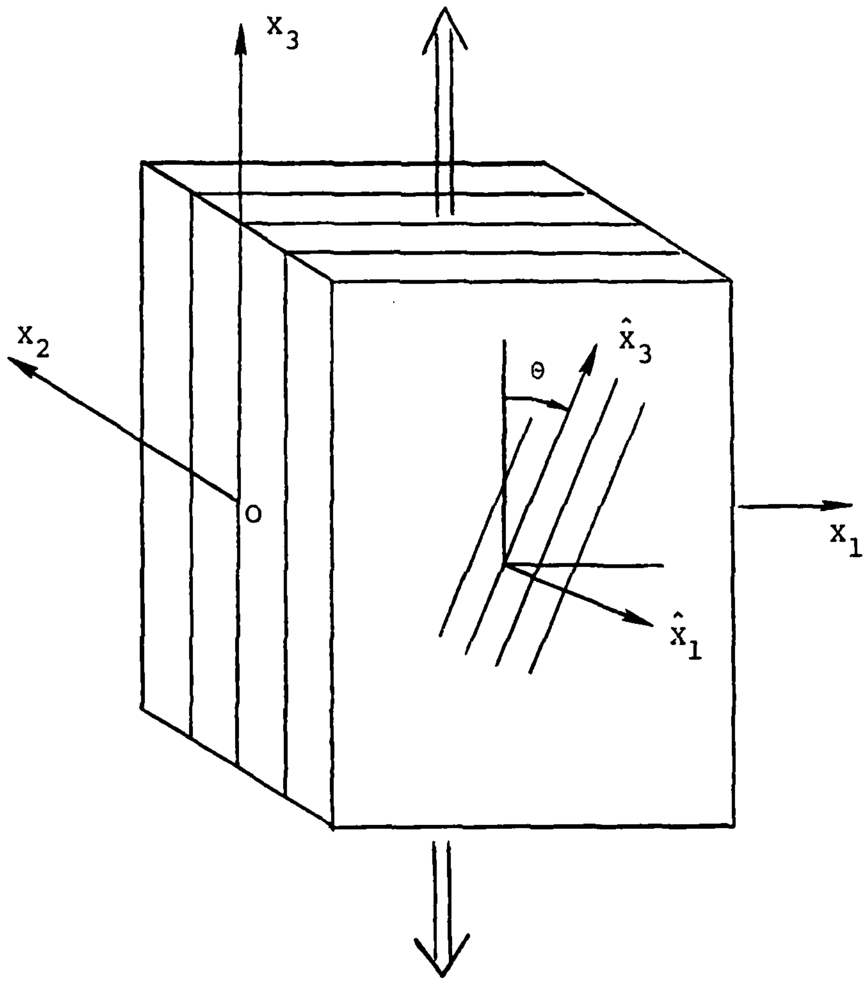

Fig. 1 Geometry of an angle-ply laminated composite

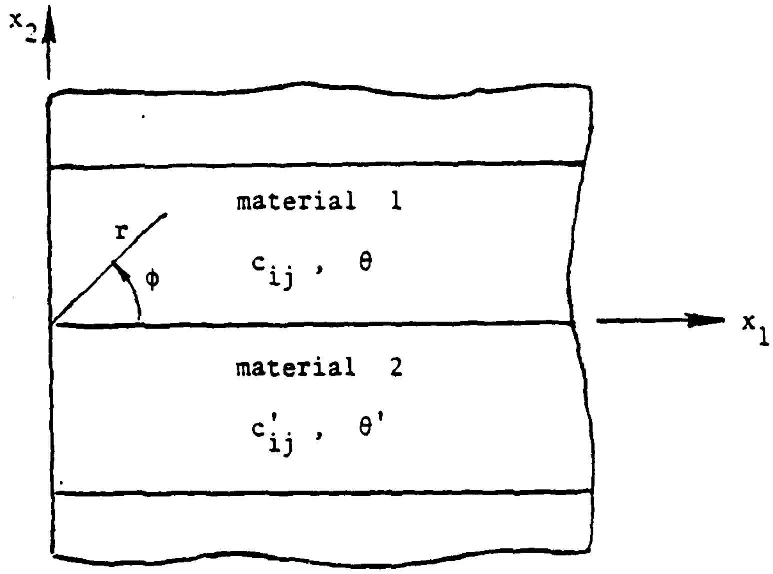

Fig. 2 A free-edge between two adjacent layers (θ/θ')

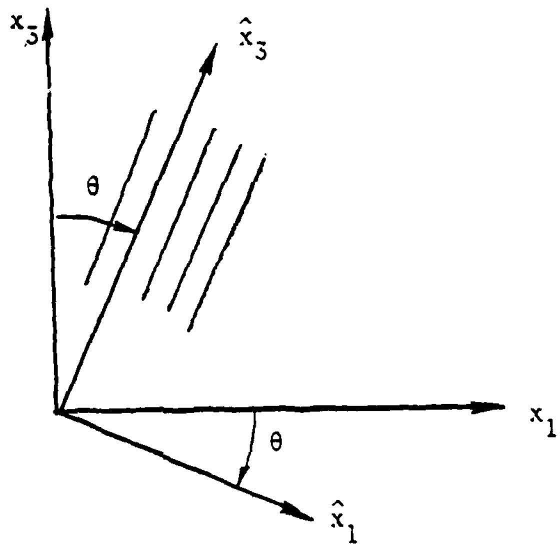

Fig. 3 Principal directions of an angle-ply laminate

Chapter II

TWO DIMENSIONAL FREE-EDGE SINGULARITY ANALYSIS

2.1 FORMULATION OF THE MATRIX $\underset{\sim}{K}$

In order to study the stress at a free-edge of an interface of a laminated composite, the origin of the (x_1, x_2, x_3) axes is placed at one of the free-edge points, as shown in Fig. 2. The (x_1, x_3) plane is along the interface, while the (x_2, x_3) plane is the free-edge surface. In this chapter, we assume that u_i and hence ε_{ij} and σ_{ij} are independent of x_3. Let

$$u_i = \upsilon_i f(Z) \tag{2.1}$$

$$\sigma_{ij} = \tau_{ij} df(Z)/dZ \tag{2.2}$$

$$Z = x_1 + px_2 \tag{2.3}$$

where p, υ_i and τ_{ij} are constants, and f(Z) is a function of Z which will be specified later. To determine the eigenvalue p and the eigenvectors υ_i and τ_{ij} in Eqs. (2.1, 2.2, 2.3), these equations are substituted into Eqs. (1.1, 1.2, 1.4). The resulting equations for p, υ_i, τ_{ij} are, [22]

$$D_{ik}\upsilon_k = 0 \tag{2.4a}$$

$$\tau_{ij} = (c_{ijk1} + pc_{ijk2})\upsilon_k \tag{2.4b}$$

where

$$D_{ik} = Q_{ik} + p(R_{ik} + R_{ki}) + p^2 T_{ik} \tag{2.4c}$$

and

$$Q_{ik} = c_{i1k1}, \quad R_{ik} = c_{i1k2}, \quad T_{ik} = c_{i2k2} \tag{2.4d}$$

For a non-trivial solution of υ_i, it follows from Eq. (2.4a) that the determinant of D_{ik} must vanish. This results in a sextic equation for p. Since the eigenvalues p are all non-real [22, 14, 18], there are three pairs of complex conjugates for p and three pairs of associated eigenvectors υ_i. For isotropic materials, all p's have the value $\pm i$.

To analyze the singular nature of the stresses at the origin, the function f(Z) is chosen to be

$$f(Z) = Z^{1+\delta}/(1+\delta) \tag{2.5}$$

where δ is a constant. Equations (2.1, 2.2) for displacement and stress can then be written as

$$u_i = \Sigma \{A_L \upsilon_{i,L} Z_L^{1+\delta} + B_L \bar{\upsilon}_{i,L} \bar{Z}_L^{1+\delta}\}/(1+\delta) \tag{2.6}$$

$$\sigma_{ij} = \Sigma \{A_L \tau_{ij,L} Z_L^{\delta} + B_L \bar{\tau}_{ij,L} \bar{Z}_L^{\delta}\} \tag{2.7}$$

where an overbar denotes the complex conjugate; A_L and B_L are complex constants, and the subscript L identifies the three pairs of eigenvalues. Unless otherwise indicated, Σ in Eqs. (2.6, 2.7) and in the sequel, stands for summation over L from L = 1 to 3. Using the polar coordinates (r, ϕ), Fig. 2, Z may be rewritten as [11, 12]

$$Z = x_1 + px_2 = r\zeta \tag{2.8}$$

where

$$\zeta = \cos\phi + p\sin\phi \tag{2.9}$$

Equations (2.6, 2.7) can then be written as

$$u_i = r^{1+\delta} \, \Sigma \, \{A_L \upsilon_{i,L} \zeta_L^{\;1+\delta} + B_L \bar{\upsilon}_{i,L} \bar{\zeta}_L^{\;1+\delta}\}/(1 + \delta) \tag{2.10}$$

$$\sigma_{ij} = r^{\delta} \, \Sigma \, \{A_L \tau_{ij,L} \zeta_L^{\;\delta} + B_L \bar{\tau}_{ij,L} \bar{\zeta}_L^{\;\delta}\} \tag{2.11}$$

Similar equations may be written for the material with elasticity constants c_{ij}' by adding a prime to all quantities except r, ϕ and δ. We see from Eq. (2.11) that if the real part of δ is negative, σ_{ij} is singular at $r = 0$. By applying the stress free boundary conditions

$$\sigma_1 = \sigma_5 = \sigma_6 = 0 \tag{2.12}$$

at $\phi = \pm\pi/2$ and the interface continuity conditions at $\phi = 0$

$$[u_1] = [u_2] = [u_3] = 0 \tag{2.13a}$$

$$[\sigma_2] = [\sigma_4] = [\sigma_6] = 0 \tag{2.13b}$$

where $[f] = f - f'$ represents the difference in f values across the interface, one obtains 12 linear homogeneous equations for A_L, B_L, A_L', B_L' which can be written as

$$\underset{\sim}{K}_c(\delta)\underset{\sim}{q} = \underset{\sim}{0} \tag{2.14}$$

where $\underset{\sim}{K}_c$ is a complex valued square matrix whose elements depend on δ, and $\underset{\sim}{q}$ is a column matrix whose elements are A_L, B_L, A_L', B_L', $(L = 1, 2, 3)$. If δ, which can be real of complex, is a root of the determinant

$$\|\underset{\sim}{K}_c(\delta)\| = 0 \tag{2.15}$$

then a nontrivial solution exists for $\underset{\sim}{q}$, and hence for the stress and displacement.

When the root δ of Eq. (2.15) is real we may choose

$$B_L = \bar{A}_L = (a_L + i\breve{a}_L)/2 \tag{2.16}$$

where a_L and $\tilde{a}_L$ are real. Equations (2.10) and (2.11) then have the real expressions

$$u_i = r^{1+\delta} \Sigma \{a_L \mathrm{Re}(\upsilon_{i,L}\zeta_L^{1+\delta}) + \tilde{a}_L \mathrm{Im}(\upsilon_{i,L}\zeta_L^{1+\delta})\}/(1 + \delta) \tag{2.17}$$

$$\sigma_{ij} = r^{\delta} \Sigma \{a_L \mathrm{Re}(\tau_{ij,L}\zeta_L^{\delta}) + \tilde{a}_L \mathrm{Im}(\tau_{ij,L}\zeta_L^{\delta})\} \tag{2.18}$$

where Re and Im stand for real and imaginary, respectively. Equation (2.14) is then replaced by

$$\underset{\sim}{K}(\delta)\underset{\sim}{a} = \underset{\sim}{0} \tag{2.19}$$

where $\underset{\sim}{K}$ is now a real valued square matrix and $\underset{\sim}{a}$ is a real column matrix whose elements are a_L, $\tilde{a}_L$, a_L' and $\tilde{a}_L'$, $(L = 1, 2, 3)$. The real root δ is then obtained from

$$\|\underset{\sim}{K}(\delta)\| = 0 \tag{2.20}$$

Before we present a numerical procedure for finding a complex root from Eq. (2.15), we point out that $\delta = 0$ is always a root of Eq. (2.15) and Eq. (2.20). It should also be pointed out that the formulation here tacitly assumed that the eigenvalues p_L of the elasticity constants, Eq. (2.4a) are distinct. For degenerate cases in which p_L has a multiple root, Eqs. (2.6, 2.7) have different expressions. The correct expressions for Eqs. (2.6, 2.7) when p_L has a multiple root were given in [11] when f(Z) assumes the special form of Eq. (2.5) and in [23] when f(Z) is arbitrary.

2.2 PROCEDURE TO FIND THE ROOTS OF $\|\underset{\sim}{K}_c\|$

To find the roots δ of Eq. (2.15), which may be complex, the following method was used. Let

$$\delta = \xi + i\eta \tag{2.21}$$

$$\|\underset{\sim}{K}_c\| = u + iv \tag{2.22}$$

The complex plane δ is divided into square regions of prescribed size. The nodal points of a typical square region have the coordinates ξ_ℓ, η_ℓ (ℓ = 1, 2, 3, 4), as shown in Fig. 4. The value of $\|\underset{\sim}{K}_c\|$ at each node of the square is

$$u_\ell = \mathrm{Re}(\|\underset{\sim}{K}_c\|)|_{\delta} = \xi_\ell + i\eta_\ell \tag{2.23}$$

$$v_\ell = \mathrm{Im}(\|\underset{\sim}{K}_c\|)|_{\delta} = \xi_\ell + i\eta_\ell \tag{2.24}$$

If the square contains the curve u = 0, then either the value at one of the nodes will be zero (in this case the curve intersects the node), or the sign of u will change between at least one pair of adjacent nodes, Fig 5. A possible exception to this is the case shown in Fig 6, which may be overcome by initially selecting a small enough square. To test if the line u = 0 is present, the computer program checks the nodal values u_ℓ for zero, and also checks for a change of sign. If either of these conditions is satisfied, the program repeats the same procedure to determine if the v = 0 curve exists in the square. When the program determines that both curves are contained in the square, then the square is divided into four smaller squares, each of which is analyzed in the same way, and again subdivided if necessary, Fig. 5. A square which is found to contain only one of the curves, or neither of the curves is discarded. The procedure continues until either no squares remain, or until the dimension of the square is smaller than a specified error parameter. If this parameter is small enough, then the fact that the two curves exist in the square, would suggest that they intersect, and that a root exists inside that square. A flow chart of the subroutine can be found in Fig. 7.

It should be pointed out that for a very small square which contains the root δ, $\|\underset{\sim}{K}_c\|$ can be quite large at the four nodes of the square. For example, Table 1 lists the values of δ and $\|\underset{\sim}{K}_c\|$ found by the above method with the dimension of the square being reduced to 10^{-9}. The values of $\|\underset{\sim}{K}_c\|$ at the four nodes are in the order of 10^5.

2.3 THE ROOTS δ

Two different laminated composites are used for the numerical calculations. Each layer of a composite is assumed to be made of the same orthotropic material. The orientation of the axes of symmetry $(\hat{x}_1, x_2, \hat{x}_3)$, however, differs from layer to layer. The following engineering constants for the layers in the two composites are taken from [6, 24], respectively.

Composite W

(Typical high modulus graphite-epoxy, [6])

$$\left.\begin{aligned} &E_1 = E_2 = 2.1 \times 10^6 \text{ psi} \\ &E_3 = 20 \times 10^6 \text{ psi} \\ &G_{12} = G_{23} = G_{31} = .85 \times 10^6 \text{ psi} \\ &\nu_{21} = \nu_{31} = \nu_{32} = .21 \end{aligned}\right\} \tag{2.25}$$

Composite T

(T300/5208 graphite epoxy, [24])

$$\left.\begin{aligned} &E_1 = E_2 = 1.54 \times 10^6 \text{ psi} \\ &E_3 = 22 \times 10^6 \text{ psi} \\ &G_{12} = G_{23} = G_{31} = .81 \times 10^6 \text{ psi} \\ &\nu_{21} = \nu_{31} = \nu_{32} = .28 \end{aligned}\right\} \qquad (2.26)$$

Using Eq. (2.25) or Eq. (2.26), $\hat{s}_{ij}$ is obtained from Eq. (1.12), while $\hat{c}_{ij}$ is computed from $\hat{s}_{ij}$ using the relations derived in [25, 26]. s_{ij}, s_{ij}', c_{ij} and c_{ij}' associated with various θ and θ' are then determined, from Eq. (1.13) and from equations similar to (1.13), [22]. Equations (2.4) provide the eigenvalues p_L, (L = 1, 2, 3) and the associated eigenvectors $\upsilon_{i,L}$ and $\tau_{ij,L}$. For Composite W, all three eigenvalues p_L are purely imaginary for any angle-ply θ, [22], while for Composite T two of the three eigenvalues are complex for $|\theta|$ less than 71.5377°. By substituting Eqs. (2.10, 2.11) into the stress free boundary conditions and the interface continuity conditions Eqs. (2.12, 2.13), one obtains a system of 12 linear homogeneous equations for the constants A_L, B_L, A_L', B_L', Eq. (2.14). The roots of the determinant of this system are found by the method described in the previous section. An area bounded by ($-1 < \xi < 3$, $0 < \eta < 3$) was checked. The roots found for various (θ/θ') combinations are listed in Tables 4 and 5. Double precision was used in the calculations but we have rounded the roots in Tables 4 and 5 to four digits. Both

complex and real roots were found. Since complex conjugates of these values are also roots of $\|K_c\|$, it was not necessary to search the $\eta <$ 0 area.

Most interesting of these roots are the positive integer values of δ, which seem to consistently appear for all (θ/θ') combinations for both composites. There appears to be a negative real root for δ, but there are no other complex roots with a negative real part. Since the negative δ is the one contributing to the singular stress, we list in Tables 2 and 3 the negative δ for various combinations of (θ/θ') angles. Also, we present in Figs. 8 and 9 the negative δ for all possible combinations of (θ/θ') angles. Curves of constant δ are given only in one quarter of the (θ, θ') plane since the curves in the remaining three quarters are a repetition of the curves shown. The negative δ values also appear to be simple roots of Eq. (2.20), and hence a of Eq. (2.19) is unique up to a multiplicative constant, say k^*. By substituting a of (2.19) into (2.18), we may write Eq. (2.18) as

$$\sigma_{ij} = k^* r^\delta \sigma_{ij}{}^* \qquad (2.27)$$

where $\sigma_{ij}{}^*$ depends on ϕ. It should be pointed out that if $\delta < 0$ is is a double root of Eq. (2.15) one would have, besides the r^δ singularity, a singularity of the form $r^\delta(\ln r)$, [11].

The analysis presented here provides the order of singularity δ and $\sigma_{ij}{}^*$. k^*, which may be identified with the stress intensity factor if elements of $\sigma_{ij}{}^*$ are normalized, can be determined only by solving the global boundary value problem. For instance, one may use a finite element scheme in which a special element, whose stress is given by Eq.

(2.27), is introduced at the free-edge. If k^* associated with a free-edge point happens to be zero after solving the global boundary value problem, there is no singularity at that particular free-edge point. Therefore, <u>a singularity at the free-edge point in plane strain deformation is not certain until the global problem is solved</u>.

TABLE 1

Values at the Nodes of a Square for Composite W, (60/-60)

Node	$\xi_\ell + i\eta_\ell = \delta$	$u_\ell + iv_\ell = \|\mathbb{K}_c\|$
1	2.94158674804 + 1.775502774707 i	204801.982842 + 47335.881916 i
2	2.94158674902 + 1.775502774707 i	-64582.9527519 - 285394.704755 i
3	2.94158674902 + 1.775502774804 i	268147.7157535 - 554779.491707 i
4	2.94158674804 + 1.775502774804 i	537532.4431132 - 222048.858434 i

TABLE 2

Negative Real Roots δ for r^{δ} Terms at the Free-Edge in Composite W

θ	δ		
	$\theta' = 0$	$\theta' = 90°$	$\theta' = -\theta$
0°	------	-3.3388×10^{-2}	------
15°	-1.3528×10^{-4}	-3.2814×10^{-2}	-6.4322×10^{-4}
30°	-2.6286×10^{-3}	-2.8682×10^{-2}	-1.1658×10^{-2}
45°	-9.6461×10^{-3}	-2.0575×10^{-2}	-2.5575×10^{-2}
60°	-1.9866×10^{-2}	-1.0519×10^{-2}	-2.3346×10^{-2}
75°	-2.9388×10^{-2}	-2.6785×10^{-3}	-8.9444×10^{-3}
90°	-3.3388×10^{-2}	------	------

TABLE 3

Negative Real Roots δ for r^{δ} Terms at the Free-Edge in Composite T

θ	δ		
	$\theta' = 0$	$\theta' = 90°$	$\theta' = -\theta$
0°	------	-5.4148×10^{-2}	------
15°	-5.5587×10^{-4}	-5.2192×10^{-2}	-2.5363×10^{-3}
30°	-5.8892×10^{-3}	-4.4295×10^{-2}	-2.2505×10^{-2}
45°	-1.8423×10^{-2}	-3.0453×10^{-2}	-3.8593×10^{-2}
60°	-3.4756×10^{-2}	-1.4602×10^{-2}	-3.1271×10^{-2}
75°	-4.8646×10^{-2}	-3.4296×10^{-3}	-1.1217×10^{-2}
90°	-5.4148×10^{-2}	------	------

TABLE 4

Complex Roots* δ for r^{δ} Terms at the Free-Edge in Composite W

(θ/θ')	δ_1	δ_2	δ_3	δ_4	δ_5
15/-15	-.001	.997 + .042 i	1.974 + .187 i	2.961 + .261 i	
0/15	-.000	.999 + .023 i	1.995 + .085 i	2.991 + .134 i	
90/15	-.033	1.411 + .387 i	1.660 + .688 i	2.835 + 1.758 i	
60/-60	-.023	.831 + .271 i	1.482 + .773 i	2.149 + 1.246 i	2.942 + 1.755 i
0/60	-.020	1.009 + .108 i	1.747 + .659 i		
90/60	-.011	.942 + .201 i	1.917 + .681 i	2.769 + 1.243 i	

* zero and positive integers are also roots for δ

TABLE 5

Complex Roots* δ for r^{δ} Terms at the Free-Edge in Composite T

(θ/θ')	δ_1	δ_2	δ_3	δ_4	δ_5
15/-15	-.003	.990 + .079 i	1.944 + .294 i	2.933 + .382 i	
0/15	-.001	.997 + .044 i	1.989 + .135 i	2.984 + .194 i	
90/15	-.052	.853	1.405 + .572 i	1.650 + .665 i	2.846 + 1.819 i
60/-60	-.031	.787 + .314 i	1.450 + .817 i	2.093 + 1.336 i	2.940 + 1.806 i
0/60	-.034	1.023	1.716 + .703 i		
90/60	-.015	1.781	.921 + .243 i	1.940 + .738 i	2.728 + 1.243 i

* zero and positive integers are also roots for δ

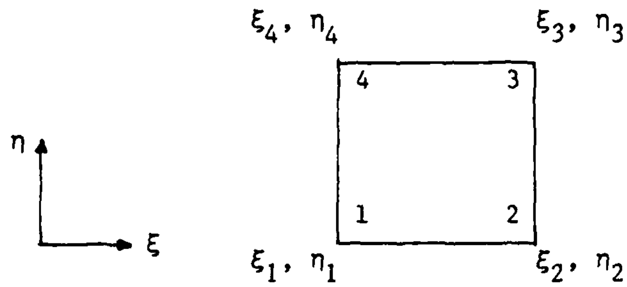

Fig. 4 Typical square to be checked for roots.

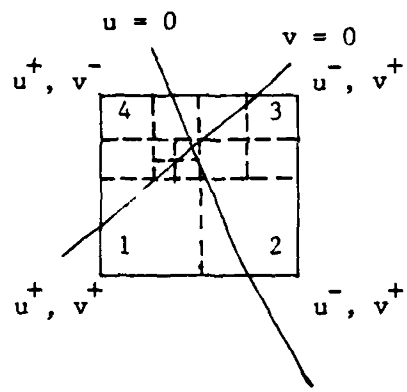

Fig. 5 Change of sign between adjacent nodes of a square.

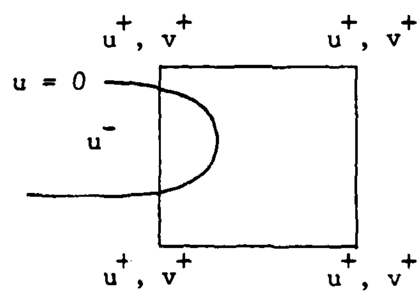

Fig. 6 A square for which u = 0 curve will not be detected.

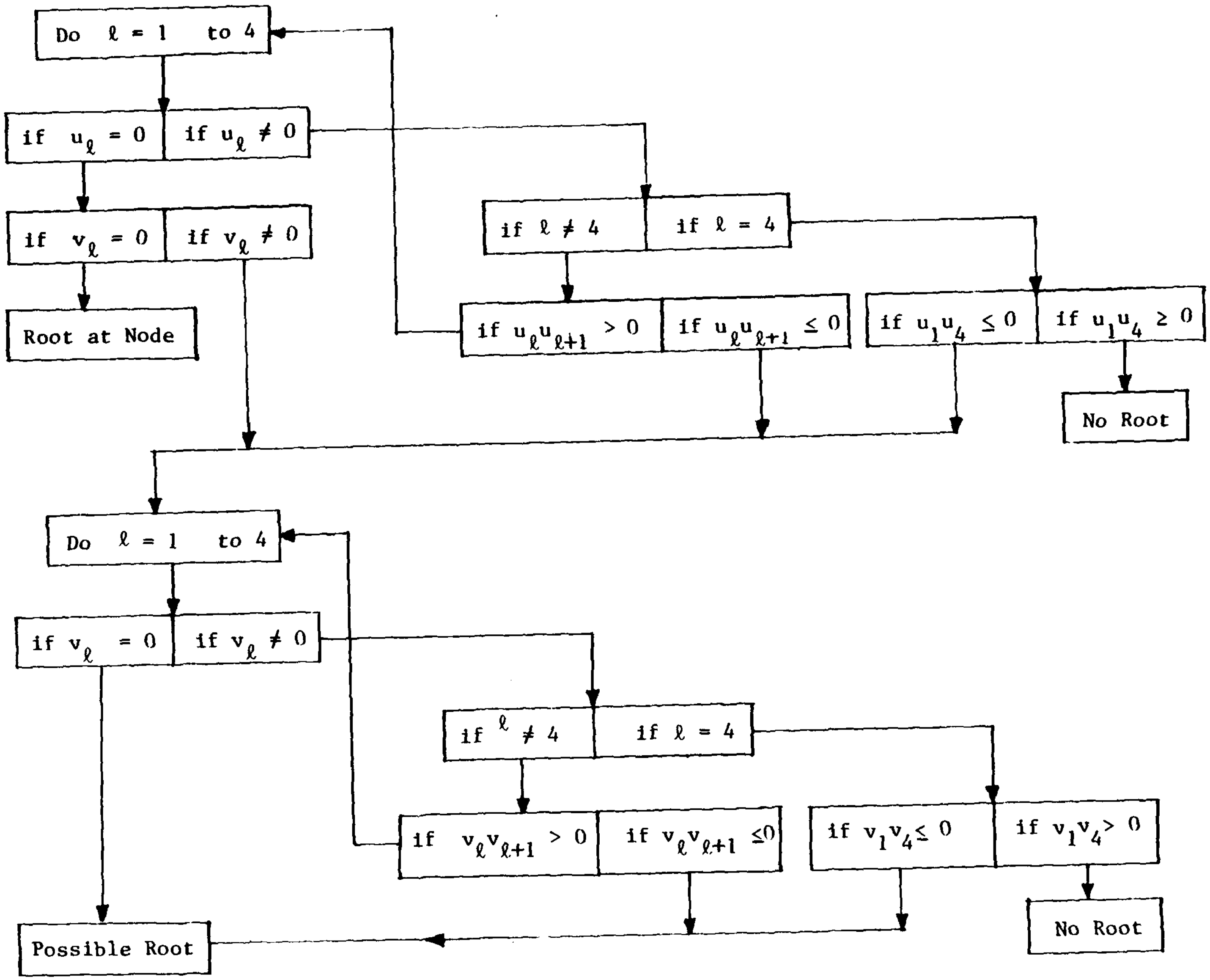

Fig. 7

Flow chart for computer subroutine that checks a square for possible complex roots

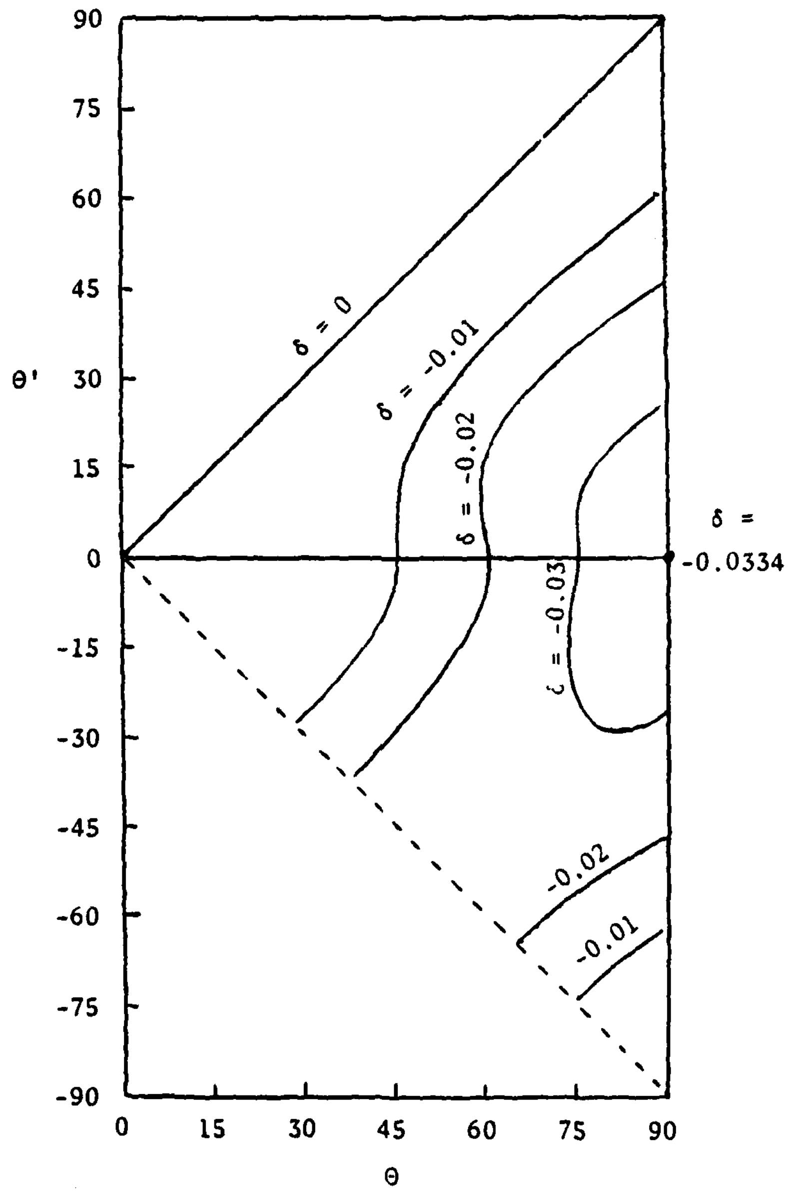

Fig. 8 δ of the r^{δ} singularity at the free-edge for composite W

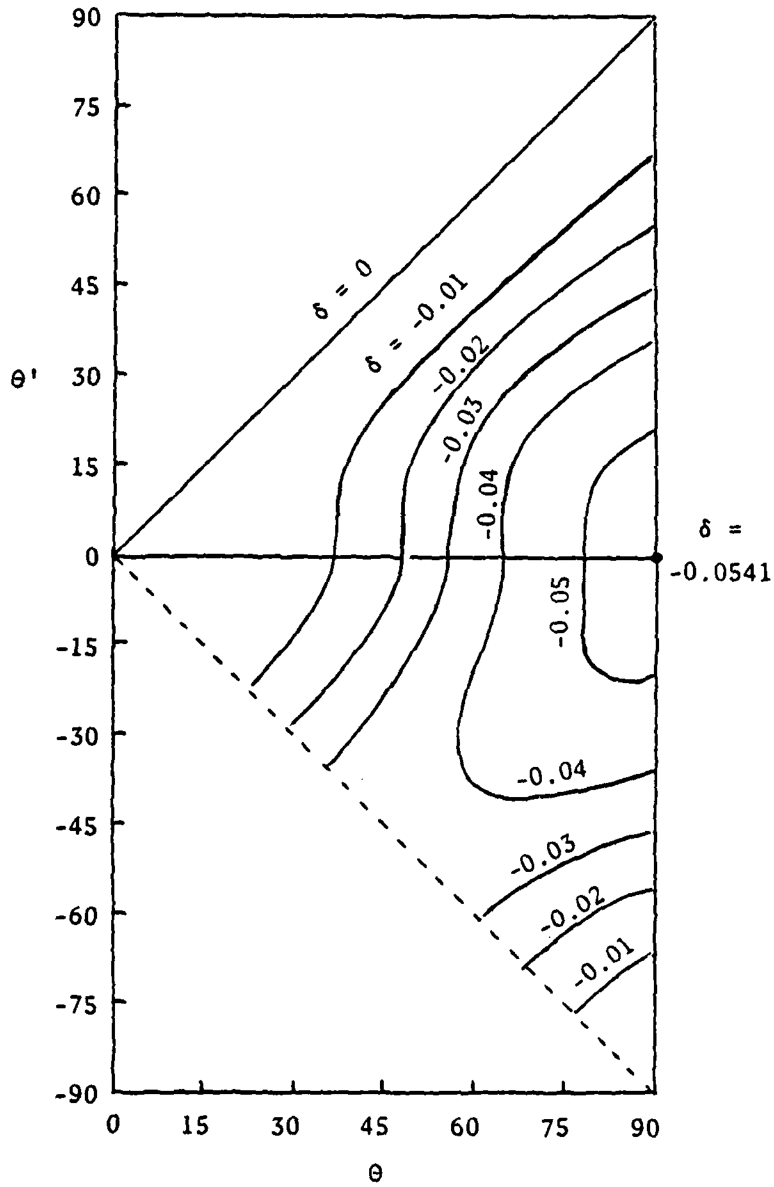

Fig. 9 δ of the r^{δ} singularity at the free-edge for composite T

Chapter III

THREE-DIMENSIONAL DEFORMATION

3.1 INTRODUCTION

The analyses presented in Chapter II assumed that the displacement u_i, and hence the strain ε_{ij} and the stress σ_{ij}, are functions of x_1 and x_2 only. In other words, we assumed a two-dimensional plane strain deformation. If the deformation is three-dimensional, u_i would depend on x_3 as well. Before we derive a first order approximation for three-dimensional deformations, we point out that the analyses presented in Chapter II remain valid for certain classes of displacements which depend on x_3.

Firstly, we may superimpose the displacement given in Eq. (2.1) by

$$u_i = U_i + \omega_{ij}x_j \tag{3.1}$$

where U_i and ω_{ij} are constants and ω_{ij} is an antisymmetric matrix. The displacement given by Eq. (3.1) represents a rigid body translation and rotation, and hence contributes nothing to the strain and stress. Thus, even though x_3 is present in Eq. (3.1), the analyses in Chapter II remain valid when Eq. (3.1) is superimposed on Eq. (2.1).

Secondly, we will show that the homogeneous strain produced by the displacement field

$$\left.\begin{aligned} u_1 &= c_1x_3 \\ u_2 &= c_2x_3 \\ u_3 &= 0 \end{aligned}\right\} \tag{3.2}$$

where c_1 and c_2 are constants can also be produced by the formulation in Chapter II by letting $\delta = 0$. Indeed, when we put $\delta = 0$ in Eq. (2.6), u_i is linear in x_1 and x_2, and hence ε_{ij} are constants. To insure that u_i assume real values with $\delta = 0$, we let $B_L = \bar{A}_L$, ($L = 1, 2, 3$). Since A_L are complex, we have six real arbitrary constants. By introducing another six real arbitrary constants a_i, b_i, c_i, ($i = 1, 2$), we may rewrite Eq. (2.6) for $\delta = 0$ as

$$\left.\begin{aligned} u_1 &= a_1x_1 + a_2x_2 \\ u_2 &= b_1x_1 + b_2x_2 \\ u_3 &= c_1x_1 + c_2x_2 \end{aligned}\right\} \tag{3.3}$$

If a_i, b_i, c_i are all non-zero, Eq. (3.3) provides all six strain components except ε_{33}. Now, consider the following special case of Eq. (3.3):

$$u_1 = u_2 = 0, \qquad u_3 = c_1x_1 + c_2x_2 \tag{3.4}$$

It is not difficult to show that the strain obtained from Eq. (3.2) and Eq. (3.4) is identical. Thus the deformation, Eq. (3.2), which depends on x_3 is a special case of the deformations considered in Chapter II.

3.2 FIRST ORDER EXPANSION

Let the displacement u_i near the origin in Fig. 2 be a function of x_1, x_2 and x_3. For points near the $x_3 = 0$ plane, we expand u_i in the following series

$$u_i = u_i^{(0)} + u_i^{(1)} x_3 + u_i^{(2)} x_3^2/2 + \ldots \tag{3.5}$$

where $u_i^{(k)}$, $(k = 0, 1, 2, \ldots)$ are functions of x_1 and x_2 only. The strain ε_{ij} is, by Eq. (1.1)

$$\begin{aligned} \varepsilon_{ij} = {} & (u_{i,j}^{(0)} + u_{j,i}^{(0)})/2 + (u_{i,j}^{(1)} + u_{j,i}^{(1)})x_3/2 + \ldots \\ & + (u_i^{(1)}\delta_{3j} + u_j^{(1)}\delta_{3i})/2 + (u_i^{(2)}\delta_{3j} + u_j^{(2)}\delta_{3i})x_3/2 \\ & + \ldots \end{aligned} \tag{3.6}$$

where δ_{ij} is the Kronecker delta. We assume that the displacement u_i is continuous and bounded. Then $u_i^{(k)}$ are continuous and bounded. In Eq. (3.6), the only terms which may contribute to a singularity in ε_{ij} must therefore come from the derivatives of $u_i^{(k)}$, i.e. from the terms $u_{i,j}^{(k)}$ and $u_{j,i}^{(k)}$. We also assume that the order of singularity δ at $x_1 = x_2 = 0$, if it exists, is continuous in x_3. This means that the order of singularity of the term $u_{i,j}^{(k)}$, $(k = 1, 2 \ldots)$ cannot be stronger than the order of the singularity of $u_{ij}^{(0)}$. Therefore, to the first order of approximation as $(x_1, x_2, x_3) \rightarrow (0, 0, 0)$ the strains are

$$\varepsilon_{ij} = (u_{i,j}{}^{(0)} + u_{j,i}{}^{(0)})/2 + (u_i{}^{(1)}\delta_{3j} + u_j{}^{(1)}\delta_{3i})/2 \qquad (3.7)$$

The terms in the first parentheses provide singular (infinite) strains while the terms in the second parantheses yield a finite value because $u_i{}^{(1)}$ is continuous and bounded. Let

$$u_i{}^{(1)} = c_i + \ldots \qquad (3.8)$$

where c_i, $(i = 1, 2, 3)$ are constants and the dots represent terms which vanish when $(x_1, x_2) \to (0, 0)$. Hence, to the first order of approximation, we may write Eqs. (3.5) as

$$u_i = u_i{}^{(0)} + c_i x_3 + \ldots \qquad (3.9)$$

The $u_i{}^{(0)}$ term is the two-dimensional deformation and is identical to the right hand side of Eq. (2.1) while the $c_i x_3$ term is the first order approximation. However, as we discussed in Eq. (3.2), the deformation for $u_1 = c_1 x_3$, $u_2 = c_2 x_3$ can be included in the two-dimensional deformations. We may therefore omit c_1 and c_2 without loss of generality, and write Eq. (3.9) as

$$u_i = \upsilon_i f(z) + \varepsilon_3 \delta_{i3} x_3 \qquad (3.10)$$

where $\varepsilon_3 = c_3$ is the strain in the x_3- direction.

Thus, to the first order of approximation, the stress singularity at a wedge corner in three-dimensional deformations can be analyzed by considering the two-dimensional plane strain problem superimposed by a uniform extension in the x_3-direction. In the next chapter,

Chapter IV, we will consider stress singularities at a free-edge point. In Chapter V the singularities at a contact-edge point will be discussed.

The seemingly innocent appearance of the uniform extension term, as we will see later, makes the stress at the free-edge point inherently singular for certain composites. Moreover, the singularity is logarithmic.

Chapter IV

THREE-DIMENSIONAL FREE-EDGE SINGULARITY ANALYSIS

The problem studied here is the same as shown in Fig. 2 except that now uniform extension ε_3 is added; thus making this problem a first order approximation in three dimensions. We will first assume that a homogeneous (uniform) stress solution exists for a specified ε_3. Two methods will be used to analyze this problem. They give the same solution for the cases of cross-ply and $(\theta/-\theta)$ composites. For composites which are neither cross-ply nor $(\theta/-\theta)$, a uniform stress solution due to the extension ε_3 does not exist.* For these composites, we find a solution in which (ln r) terms appear in the expressions for stress and displacement. These will, of course, lead to singularities at $r = 0$. Singularities of the type r^{δ} $(-1 < \mathrm{Re}(\delta) < 0)$ found in the analysis of the two dimensional plane strain problem, Chapter II, may still exist, and should be superimposed on the solution obtained here.

4.1 HOMOGENEOUS STRESS SOLUTION: METHOD 1

In this section we assume that a homogeneous stress solution exists due to a uniform extension in the x_3-direction. Hence the stress σ_{ij} and the strain ε_{ij} are independent of x_1 and x_2. By using the notations discussed in Section 1.2 along with Eq. (1.8, 1.9, 1.10) one can obtain

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* For an exception see page 45

the relations between the strain ε_{ij} and the stress σ_{ij} in the following form

$$\varepsilon_1 = s_{11}\sigma_1 + s_{12}\sigma_2 + s_{13}\sigma_3 + s_{14}\sigma_4 + s_{15}\sigma_5 + s_{16}\sigma_6 \quad (4.1)$$

$$\varepsilon_3 = s_{31}\sigma_1 + s_{32}\sigma_2 + s_{33}\sigma_3 + s_{34}\sigma_4 + s_{35}\sigma_5 + s_{36}\sigma_6 \quad (4.2)$$

$$\varepsilon_5 = s_{51}\sigma_1 + s_{52}\sigma_2 + s_{53}\sigma_3 + s_{54}\sigma_4 + s_{55}\sigma_5 + s_{56}\sigma_6 \quad (4.3)$$

where the expressions for ε_2, ε_4 and ε_6 are not needed in the following analysis, and hence, are omitted here. Since the material is symmetric with respect to the (x_1, x_3) plane, [25, 26]

$$s_{14} = s_{16} = s_{24} = s_{26} = s_{34} = s_{36} = s_{54} = s_{56} = 0 \quad (4.4)$$

By applying the above material symmetry properties, along with the stress free boundary conditions at $\phi = \pm\pi/2$, Eq. (2.12), one obtains

$$\varepsilon_1 = s_{12}\sigma_2 + s_{13}\sigma_3 \quad (4.5)$$

$$\varepsilon_3 = s_{32}\sigma_2 + s_{33}\sigma_3 \quad (4.6)$$

$$\varepsilon_5 = s_{52}\sigma_2 + s_{53}\sigma_3 \quad (4.7)$$

Solving for σ_3 from Eq. (4.6), and eliminating σ_3 in Eqs. (4.5, 4.7) we have

$$\sigma_3 = \varepsilon_3/s_{33} - (s_{32}/s_{33})\sigma_2 \quad (4.8)$$

$$\varepsilon_1 = \varepsilon_3 s_{13}/s_{33} + (s_{12} - s_{13}s_{32}/s_{33})\sigma_2 \quad (4.9)$$

$$\varepsilon_5 = \varepsilon_3 s_{53}/s_{33} + (s_{52} - s_{53}s_{32}/s_{33})\sigma_2 \quad (4.10)$$

By setting

$$\left.\begin{aligned} R_1 &= s_{13}/s_{33} \\ R_{12} &= s_{12} - s_{13}s_{32}/s_{33} \\ R_5 &= s_{53}/s_{33} \\ R_{52} &= s_{52} - s_{53}s_{32}/s_{33} \end{aligned}\right\} \quad (4.11)$$

Eq. (4.9) and Eq. (4.10) can be rewriten as

$$\varepsilon_1 = R_1\varepsilon_3 + R_{12}\sigma_2 \tag{4.12}$$

$$\varepsilon_5 = R_5\varepsilon_3 + R_{52}\sigma_2 \tag{4.13}$$

The interface continuity conditions of Eq. (2.13a) at $\phi = 0$ are equivalent to

$$[\varepsilon_1] = [\varepsilon_3] = [\varepsilon_5] = 0 \tag{4.14}$$

Using Eqs. (2.13b) and (4.14) Eqs. (4.12, 4.13) become, respectively,

$$[R_{12}]\sigma_2 + [R_1]\varepsilon_3 = 0 \tag{4.15}$$

$$[R_{52}]\sigma_2 + [R_5]\varepsilon_3 = 0 \tag{4.16}$$

For cross-ply composites, i.e. (0/90) composites, $R_5 = R_5' = R_{52} = R_{52}' = 0$ because we have assumed that material 1 is identical to material 2. Hence Eq. (4.16) is automatically satisfied and Eq. (4.15) yields

$$\sigma_2 = -\varepsilon_3[R_1]/[R_{12}] \tag{4.17}$$

For $(\theta/-\theta)$ composites, $R_{12} = R_{12}'$ and $R_1 = R_1'$. Equation (4.15) is automatically satisfied and Eq. (4.16) provides σ_2:

$$\sigma_2 = -\varepsilon_3[R_5]/[R_{52}] \tag{4.18}$$

For other (θ/θ') combinations, Eqs. (4.15) and (4.16) contradict each other.* This indicates that the assumed homogeneous stress solution due to a prescribed ε_3 extension does not exist for other (θ/θ') composites.

When a homogeneous solution exists, σ_2 is obtained from Eq. (4.17) or (4.18), σ_3 is from Eq. (4.8), while σ_4 is arbitrary. If we denote by $\sigma_2^{(p)}$, $\sigma_3^{(p)}$ the values of σ_2 and σ_3 obtained from

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* For an exception see page 45

Eqs. (4.17 or 4.18) and (4.8) we may write the homogeneous solution as

$$\sigma_{ij} = \varepsilon_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_2^{(p)} & 0 \\ 0 & 0 & \sigma_3^{(p)} \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.19}$$

where α_1 is an arbitrary constant. A similar equation applies to σ_{ij}'.

4.2 HOMOGENEOUS STRESS SOLUTION: METHOD 2

The preceeding approach fails to give a solution when Eqs. (4.15) and (4.16) are incompatible. The following method, which is a modified version of the method in Chapter II, provides a means to solve the cases for which a homogeneous stress solution does not exist. However, we will first apply the method to find the homogeneous stress solution when it exists, and see if this method produces the same solution as in the previous section.

To take into account the uniform extension in the x_3 direction, Eq. (2.1) is replaced by Eq. (3.10) so that

$$u_i = \upsilon_i f(Z) + \delta_{i3}\varepsilon_3 x_3 \tag{4.20}$$

$$\sigma_{ij} = \tau_{ij} df(Z)/dZ + c_{ij33}\varepsilon_3 \tag{4.21}$$

By Eqs. (1.1, 1.2) one obtains expressions for p, τ_{ij}, υ_k which are the same as those for the two dimensional plane strain problem, Eq. (2.4). Choosing f(Z) to have the form defined in Eq. (2.5), Eqs. (4.20, 4.21) may be rewritten as

$$u_i = \Sigma \{A_L \upsilon_{i,L} Z_L^{1+\delta} + B_L \bar{\upsilon}_{i,L} \bar{Z}_L^{1+\delta}\}/(1+\delta) + \varepsilon_3 \delta_{i3} x_3 \tag{4.22}$$

$$\sigma_{ij} = \Sigma \{A_L \tau_{ij,L} Z_L^{\delta} + B_L \bar{\tau}_{ij,L} \bar{Z}_L^{\delta}\} + c_{ij33}\varepsilon_3 \tag{4.23}$$

where A_L and B_L are complex constants. Equations (4.22, 4.23) differ from Eqs. (2.6, 2.7) in the last term. Since Eqs. (2.6, 2.7) can be written in the form of Eqs. (2.17, 2.18) when δ is real, we rewrite Eqs. (4.22, 4.23) for real δ as

$$u_i = r^{1+\delta} \Sigma \{a_L \mathrm{Re}(\upsilon_{i,L}\zeta_L^{1+\delta}) + \tilde{a}_L \mathrm{Im}(\upsilon_{i,L}\zeta_L^{1+\delta})\}/(1+\delta) + \varepsilon_3\delta_{i3}x_3 \qquad (4.24)$$

$$\sigma_{ij} = r^{\delta} \Sigma \{a_L \mathrm{Re}(\tau_{ij,L}\zeta_L^{\delta}) + \tilde{a}_L \mathrm{Im}(\tau_{ij,L}\zeta_L^{\delta})\} + c_{ij33}\varepsilon_3 \qquad (4.25)$$

where a_L and $\tilde{a}_L$ are real coefficients. By applying the stress free boundary conditions specified in Eq. (2.12) at $\phi = \pm\pi/2$, and the interface continuity condition at $\phi = 0$, Eq. (2.13), one obtains 12 linear equations for a_L, $\tilde{a}_L$, a_L', $\tilde{a}_L'$ which can be written as

$$r^{\delta}\underset{\sim}{K}(\delta)\underset{\sim}{a} = \varepsilon_3\underset{\sim}{b} \qquad (4.26)$$

where $\underset{\sim}{b}$ is a column matrix whose elements consist of c_{ij33} and c'_{ij33}. In the special case when $\varepsilon_3 = 0$, Eq. (4.26) reduces to Eq. (2.19).

Since the right hand side of Eq. (4.26) is independent of r, Eq. (4.26) holds only for $\delta = 0$:

$$\underset{\sim}{K}(0)\underset{\sim}{a} = \varepsilon_3\underset{\sim}{b} \qquad (4.27)$$

Assuming that (4.27) has a solution for $\underset{\sim}{a}$ whose elements are a_L, $\tilde{a}_L$, a_L', $\tilde{a}_L'$, and substitute the results into Eqs. (4.24, 4.25) with $\delta = 0$ we obtain

$$u_i = \Sigma \{a_L \mathrm{Re}(\upsilon_{i,L}Z_L) + \tilde{a}_L \mathrm{Im}(\upsilon_{i,L}Z_L)\} + \varepsilon_3\delta_{i3}x_3 \qquad (4.28)$$

$$\sigma_{ij} = \Sigma \{a_L \mathrm{Re}(\tau_{ij,L}) + \tilde{a}_L \mathrm{Im}(\tau_{ij,L})\} + c_{ij33}\varepsilon_3 \qquad (4.29)$$

where Eq. (2.8) has been used in obtaining Eq. (4.28). Equation (4.29) shows that σ_{ij} is homogeneous.

As we pointed out in Section 2.1, the determinant of $\underset{\sim}{K}(\delta)$ vanishes for $\delta = 0$. Hence, a solution to Eq. (4.27) exists if and only if, [27]

$$\underset{\sim}{\ell}^T \underset{\sim}{b} = 0 \tag{4.30}$$

where $\underset{\sim}{\ell}^T$ is a left eigenvector of $\underset{\sim}{K}(0)$:

$$\underset{\sim}{\ell}^T \underset{\sim}{K}(0) = \underset{\sim}{0} \tag{4.31}$$

It turns out that there are two left eigenvectors of $\underset{\sim}{K}(0)$. Equation (4.30) then, must be satisfied for both $\underset{\sim}{\ell}$. For the (0/90) and $(\theta/-\theta)$ composites, numerical solutions indicated that Eq. (4.30) holds, and Eq. (4.27) has a particular solution $\varepsilon_3 \underset{\sim}{a}^{(p)}$ and two arbitrary solutions $\underset{\sim}{a}^{(1)}$ and $\underset{\sim}{a}^{(2)}$. Hence

$$\underset{\sim}{a} = \varepsilon_3 \underset{\sim}{a}^{(p)} + \alpha_1 \underset{\sim}{a}^{(1)} + \alpha_2 \underset{\sim}{a}^{(2)} \tag{4.32}$$

where α_1 and α_2 are arbitrary constants. Substitution of Eq. (4.32) into Eqs. (4.28, 4.29) yields

$$u_i = \varepsilon_3 u_i^{(p)} + \alpha_1 u_i^{(1)} + \alpha_2 u_i^{(2)} \tag{4.33}$$

$$\sigma_{ij} = \varepsilon_3 \sigma_{ij}^{(p)} + \alpha_1 \sigma_{ij}^{(1)} \tag{4.34}$$

where

$$u_i^{(p)} = \Sigma \{a_L^{(p)} \mathrm{Re}(\upsilon_{i,L} Z_L) + \tilde{a}_L^{(p)} \mathrm{Im}(\upsilon_{i,L} Z_L)\} + \delta_{i3} x_3 \tag{4.35a}$$

$$u_i^{(n)} = \Sigma \{a_L^{(n)} \mathrm{Re}(\upsilon_{i,L} Z_L) + \tilde{a}_L^{(n)} \mathrm{Im}(\upsilon_{i,L} Z_L)\}, \ (n = 1, 2) \tag{4.35b}$$

$$\sigma_{ij}^{(p)} = \Sigma \{a_L^{(p)} \mathrm{Re}(\tau_{ij,L}) + \tilde{a}_L^{(p)} \mathrm{Im}(\tau_{ij,L})\} + c_{ij33} \tag{4.35c}$$

$$\sigma_{ij}^{(1)} = \Sigma \{a_L^{(1)} \mathrm{Re}(\tau_{ij,L}) + \tilde{a}_L^{(1)} \mathrm{Im}(\tau_{ij,L})\} \tag{4.35d}$$

The reason $\sigma_{ij}^{(2)}$ is absent in Eq. (4.34) is because we have chosen $\underset{\sim}{a}^{(1)}$ and $\underset{\sim}{a}^{(2)}$ such that $u_i^{(2)}$ is a rigid body rotation, and hence $\sigma_{ij}^{(2)}$ associated with $u_i^{(2)}$ is zero.

Numerical solutions for the composites considered here show that Eq. (4.34) is identical to Eq. (4.19) for (0/90) and (θ/-θ) composites. When the composite is neither (0/90) nor (θ/-θ) Eq. (4.30) does not hold, and there is no solution for a from Eq. (4.27).* In the next section we will treat this case by modifying the approach of this section.

4.3 LOGARITHMIC SINGULARITY

If no solution exists for Eq. (4.27) the assumption of uniform stress due to a uniform extension ε_3 is appparently invalid. In such a case, instead of using Eqs. (4.22, 4.23) we use the following solution:

$$u_i = \frac{\partial}{\partial\delta}\{\ \Sigma\ [A_L \upsilon_{i,L} Z_L^{1+\delta} + B_L \bar{\upsilon}_{i,L} \bar{Z}_L^{1+\delta}]/(1+\delta)\} + \varepsilon_3 \delta_{i3} x_3 \tag{4.36}$$

$$\sigma_{ij} = \frac{\partial}{\partial\delta}\{\ \Sigma\ [A_L \tau_{ij,L} Z_L^{\delta} + B_L \bar{\tau}_{ij,L} \bar{Z}_L^{\delta}]\} + c_{ij33}\varepsilon_3 \tag{4.37}$$

where A_L, B_L are now functions of δ. It can be shown that Eqs. (4.36, 4.37) satisfy Eqs. (1.1, 1.2, 1.4) with $\upsilon_{i,L}$ and $\tau_{ij,L}$ given by Eq. (2.4). Before we substitute Eqs. (4.36, 4.37) into the boundary and interface conditions, we apply the equivalence between Eqs. (4.22, 4.23) and Eqs. (4.24, 4.25) to Eqs. (4.36, 4.37). The result is

$$u_i = \frac{\partial}{\partial\delta}\{r^{1+\delta}\ \Sigma\ [a_L \mathrm{Re}(\upsilon_{i,L}\zeta_L^{1+\delta}) + \tilde{a}_L \mathrm{Im}(\upsilon_{i,L}\zeta_L^{1+\delta})]/(1+\delta)\}$$
$$+ \varepsilon_3 \delta_{i3} x_3 \tag{4.38}$$

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* For an exception see page 45

$$\sigma_{ij} = \frac{\partial}{\partial\delta} \{r^{\delta} \Sigma [a_L \mathrm{Re}(\tau_{ij,L}\zeta_L^{\delta}) + \tilde{a}_L \mathrm{Im}(\tau_{ij,L}\zeta_L^{\delta})]\} + c_{ij33}\varepsilon_3 \tag{4.39}$$

where a_L, $\tilde{a}_L$ are now functions of δ. Performing the differentiation, we obtain

$$u_i = r^{1+\delta}(\ln r + \partial/\partial\delta) \Sigma [a_L \mathrm{Re}(\upsilon_{i,L}\zeta_L^{1+\delta}) + \tilde{a}_L \mathrm{Im}(\upsilon_{i,L}\zeta_L^{1+\delta})]/(1+\delta) + \varepsilon_3\delta_{i3}x_3 \tag{4.40}$$

$$\sigma_{ij} = r^{\delta}(\ln r + \partial/\partial\delta) \Sigma [a_L \mathrm{Re}(\tau_{ij,L}\zeta_L^{\delta}) + \tilde{a}_L \mathrm{Im}(\tau_{ij,L}\zeta_L^{\delta})] + c_{ij33}\varepsilon_3 \tag{4.41}$$

Equations (4.40, 4.41) differ from Eqs. (4.24, 4.25) by a factor of $(\ln r + \partial/\partial\delta)$. If we apply Eqs. (4.40) and (4.41) to the free surface conditions, Eq. (2.12) and the interface continuity conditions, Eq. (2.13), we obtain 12 equations which can be written as (cf. Eq. (4.26))

$$r^{\delta}(\ln r + \partial/\partial\delta)\, \underset{\sim}{K}(\delta)\, \underset{\sim}{a}(\delta) = \varepsilon_3 \underset{\sim}{b} \tag{4.42}$$

where $\underset{\sim}{a}$, whose components are a_L, $\tilde{a}_L$, a_L', $\tilde{a}_L'$, is now a function of δ. This equation holds for arbitrary r if we let $\delta = 0$, *and*

$$\underset{\sim}{K}(0)\underset{\sim}{a}(0) = \underset{\sim}{0} \tag{4.43}$$

$$\frac{\partial}{\partial\delta} [\underset{\sim}{K}(\delta)\underset{\sim}{a}(\delta)]|_{\delta = 0} = \varepsilon_3 \underset{\sim}{b} \tag{4.44}$$

For simplicity, we write Eq. (4.43, 4.44) as

$$\underset{\sim}{K}\underset{\sim}{a} = \underset{\sim}{0} \tag{4.45}$$

$$(d\underset{\sim}{K}/d\delta)\underset{\sim}{a} + \underset{\sim}{K}(d\underset{\sim}{a}/d\delta) = \varepsilon_3 \underset{\sim}{b} \tag{4.46}$$

where it is understood that all quantities on the left hand side of Eqs. (4.45, 4.46) are evaluated at $\delta = 0$. Equations (4.45, 4.46) consist of 24 equations for $\underset{\sim}{a}$ and $d\underset{\sim}{a}/d\delta$. If a solution exists, substitution of $\underset{\sim}{a}$ and $d\underset{\sim}{a}/d\delta$ back into Eq. (4.41) with $\delta = 0$ provides the desired solution.

Before we discuss the solution of Eqs. (4.45, 4.46) in the next section, we write Eqs. (4.40) and (4.41) in full with $\delta = 0$:

$$u_i = (\ln r)\Sigma\,[a_L \mathrm{Re}(\upsilon_{i,L} Z_L) + \tilde{a}_L \mathrm{Im}(\upsilon_{i,L} Z_L)]$$
$$+ \Sigma\,[a_L \mathrm{Re}(\upsilon_{i,L} Z_L(\ln\zeta_L - 1)) + \tilde{a}_L \mathrm{Im}(\upsilon_{i,L} Z_L(\ln\zeta_L - 1))$$
$$+ (da_L/d\delta)\mathrm{Re}(\upsilon_{i,L} Z_L) + (d\tilde{a}_L/d\delta)\mathrm{Im}(\upsilon_{i,L} Z_L)] + \varepsilon_3 \delta_{i3} x_3 \tag{4.47}$$

$$\sigma_{ij} = (\ln r)\ \Sigma\,[a_L \mathrm{Re}(\tau_{ij,L}) + \tilde{a}_L \mathrm{Im}(\tau_{ij,L})]$$
$$+ \ \Sigma\,[a_L \mathrm{Re}(\tau_{ij,L} \ln\zeta_L) + \tilde{a}_L \mathrm{Im}(\tau_{ij,L} \ln\zeta_L)$$
$$+ (da_L/d\delta)\mathrm{Re}(\tau_{ij,L}) + (d\tilde{a}_L/d\delta)\mathrm{Im}(\tau_{ij,L})] + c_{ij33}\varepsilon_3 \tag{4.48}$$

We see that σ_{ij} has a logarithmic singularity. Again, Eq. (2.8) has been used in obtaining Eq. (4.47).

4.4 SOLUTION FOR STRESS AND DISPLACEMENT

The system of Equations (4.45, 4.46) has a unique solution for $\underset{\sim}{a}$ if, (see [10])

$$d^N \|\underset{\sim}{K}\| / d\delta^N \neq 0, \quad N = n - m, \tag{4.49}$$

where n and m are, respectively, the order and rank of $\underset{\sim}{K}$. For the composites considered here, N = 2. However, it is rather difficult to prove or disprove Eq. (4.49) analytically or numerically in view of the fact that $\underset{\sim}{K}$ is a 12 x 12 matrix. Instead, we regard Eqs. (4.45, 4.46) as a system of 24 equations for $\underset{\sim}{a}$ and $d\underset{\sim}{a}/d\delta$, and solve the system numerically. We find numerically that $\underset{\sim}{a}$ is unique while $d\underset{\sim}{a}/d\delta$ has a particular solution and two arbitrary solutions.

Noting that N = 2, one can see that $\underset{\sim}{K}$ has two right eigenvectors $\underset{\sim}{a}^{(1)}$ and $\underset{\sim}{a}^{(2)}$ such that

$$\underset{\sim}{K}\underset{\sim}{a}^{(n)} = \underset{\sim}{0}, \quad (n = 1, 2) \tag{4.50}$$

If $\underset{\sim}{a}$ is the unique solution of both Eqs. (4.45) and (4.46), it must also be a solution of Eq. (4.45), and hence $\underset{\sim}{a}$ is proportional to a right eigenvector. Without loss of generality, let $\underset{\sim}{a}^{(1)}$ be the eigenvector to which $\underset{\sim}{a}$ is proportional, i.e.,

$$\underset{\sim}{a} = k\varepsilon_3 \underset{\sim}{a}^{(1)} \tag{4.51}$$

where k is uniquely determined if $\underset{\sim}{a}^{(1)}$ is properly normalized. The fact that $d\underset{\sim}{a}/d\delta$ has two arbitrary solutions is obvious from Eq. (4.46) because the coefficient of $d\underset{\sim}{a}/d\delta$ is $\underset{\sim}{K}$ which is singular of order 2. If $\varepsilon_3(d\underset{\sim}{a}/d\delta)^{(p)}$ is a particular solution of $d\underset{\sim}{a}/d\delta$, we have

$$d\underset{\sim}{a}/d\delta = \varepsilon_3(d\underset{\sim}{a}/d\delta)^{(p)} + \alpha_1\underset{\sim}{a}^{(1)} + \alpha_2\underset{\sim}{a}^{(2)} \tag{4.52}$$

where α_1, α_2 are arbitrary constants. With Eqs. (4.51, 4.52), Eqs. (4.47, 4.48) can be rewritten as

$$u_i = k\varepsilon_3[(\ln r)u_i^{(1)} + u_i^{(\phi)}] + \varepsilon_3 u_i^{(p)} + \alpha_1 u_i^{(1)} + \alpha_2 u_i^{(2)} \tag{4.53}$$

$$\sigma_{ij} = k\varepsilon_3[(\ln r)\sigma_{ij}^{(1)} + \sigma_{ij}^{(\phi)}] + \varepsilon_3\, \sigma_{ij}^{(p)} + \alpha_1\sigma_{ij}^{(1)} \tag{4.54}$$

where $u_i^{(1)}$, $u_i^{(2)}$, and $\sigma_{ij}^{(1)}$ are identical to the ones defined in Eqs. (4.35b, 4.35d), while

$$u_i^{(p)} = \Sigma\,\{(da_L/d\delta)^{(p)}\mathrm{Re}(\upsilon_{i,L}Z_L) + (d\tilde{a}_L/d\delta)^{(p)}\mathrm{Im}(\upsilon_{i,L}Z_L)\} + \delta_{i3}x_3 \tag{4.55a}$$

$$u_i^{(\phi)} = \Sigma\,\{a_L^{(1)}\mathrm{Re}[\upsilon_{i,L}Z_L(\ln\zeta_L - 1)] + \tilde{a}_L^{(1)}\mathrm{Im}[\upsilon_{i,L}Z_L(\ln\zeta_L - 1)]\} \tag{4.55b}$$

$$\sigma_{ij}^{(p)} = \Sigma\,\{(da_L/d\delta)^{(p)}\mathrm{Re}(\tau_{ij,L}) + (d\tilde{a}_L/d\delta)^{(p)}\mathrm{Im}(\tau_{ij,L})\} + c_{ij33} \tag{4.55c}$$

$$\sigma_{ij}^{(\phi)} = \Sigma \{a_L^{(1)} Re(\tau_{ij,L} \ln \zeta_L) + \tilde{a}_L^{(1)} Im(\tau_{ij,L} \ln \zeta_L)\} \qquad (4.55d)$$

Again the reason $\sigma_{ij}^{(2)}$ is missing in Eq. (4.54) is due to the fact that $u_i^{(2)}$ is a rigid body rotation.

Although the solution obtained here is for composites which are neither (0/90) nor ($\theta/-\theta$), application of the present solution to (0/90) or ($\theta/-\theta$) composites yields $k = 0$. Hence $\underset{\sim}{a} = \underset{\sim}{0}$ by Eq. (4.51), and the solution for $d\underset{\sim}{a}/d\delta$ from Eq. (4.46) is identical to the solution for $\underset{\sim}{a}$ in Eq. (4.27). It follows that $(d\underset{\sim}{a}/d\delta)^{(p)}$ of Eq. (4.52) is identical to $\underset{\sim}{a}^{(p)}$ of Eq. (4.32), and that $u_i^{(p)}$, $\sigma_{ij}^{(p)}$ in Eqs. (4.55a, 4.55c) and Eqs. (4.35a, 4.35c) are also identical. Thus, the solution obtained in Eqs. (4.53, 4.54) reduces to that given in Eqs. (4.33, 4.34) when the composite is (0/90) or ($\theta/-\theta$).

For composites which are neither (0/90) nor ($\theta/-\theta$), $k \neq 0$.[1] We see from Eq. (4.54) that σ_{ij} has a logarithmic singularity unless $\varepsilon_3 = 0$. Therefore, unless ε_3 happens to be zero in a three-dimensional deformation, the stress is inherently singular for composites which are neither (0/90) nor ($\theta/-\theta$).[1] Moreover, the singularity is logarithmic. Since the larger the value of k the stronger the logarithmic singularity, k may be regarded as the "logarithmic stress intensity factor". It should be pointed out that the singularity of $k^* r^\delta$, ($\delta < 0$), as analyzed in Chapter II, may still exist for all composites. However, the determination of the intensity factor k^* requires a global solution while k in Eq. (4.54) does not.

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[1] For an exception see page 45

If we replace the arbitrary constant α_1 by another arbitrary constant β_1 defined by

$$\alpha_1 = -k\varepsilon_3 \ln \beta_1 \tag{4.56}$$

Eq. (4.54) can be rewritten as

$$\sigma_{ij} = \varepsilon_3\{k \ln(r/\beta_1)\, \sigma_{ij}^{(1)} + \tilde{\sigma}_{ij}(\phi)\} \tag{4.57}$$

where

$$\tilde{\sigma}_{ij}(\phi) = k\sigma_{ij}^{(\phi)} + \sigma_{ij}^{(p)} \tag{4.58}$$

Thus β_1 must have the same physical dimension as r. We see from Eqs. (4.35c, 4.35d) that $\sigma_{ij}^{(p)}$ and $\sigma_{ij}^{(1)}$ are constants. For both composites considered here, $\sigma_{ij}^{(1)}$ has the form

$$\sigma_{ij}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.59}$$

On the other hand, we see from Eqs. (2.9, 4.55d) that $\sigma_{ij}^{(\phi)}$ depends on ϕ, (Fig. 3). Hence $\tilde{\sigma}_{ij}(\phi)$ and $\tilde{\sigma}_{ij}'(\phi)$ depend on ϕ. In Tables 6-9 we list the logarithmic stress intensity factor k and $\tilde{\sigma}_{ij}(\phi)$ and $\tilde{\sigma}_{ij}'(\phi)$ on the interface ($\phi = 0$), and on the free-edge surface ($\phi = \pm 90°$). Notice that $\tilde{\sigma}_{11} = \tilde{\sigma}_{12} = \tilde{\sigma}_{13} = 0$ at $\phi = 90°$. Similarly, $\tilde{\sigma}_{11}' = \tilde{\sigma}_{12}' = \tilde{\sigma}_{13}' = 0$ at $\phi = -90°$. Hence these components are not listed in the tables. Also, since $\tilde{\sigma}_{22}' = \tilde{\sigma}_{22}$, $\tilde{\sigma}_{21}' = \tilde{\sigma}_{21}$, $\tilde{\sigma}_{23}' = \tilde{\sigma}_{23}$ at $\phi = 0$, only $\tilde{\sigma}_{22}(0)$, $\tilde{\sigma}_{21}(0)$, $\tilde{\sigma}_{23}(0)$ are listed. All $\tilde{\sigma}_{ij}$ in Tables 6-10 have the unit of 10^6 psi.

For (0/90) or (θ/-θ) composites, k = 0, and Eq. (4.54) may be rewritten as

$$\sigma_{ij} = \alpha_1 \sigma_{ij}^{(1)} + \varepsilon_3 \tilde{\sigma}_{ij} \tag{4.60}$$

where

$$\tilde{\sigma}_{ij} = \sigma_{ij}^{(p)} \tag{4.61}$$

is now independent of ϕ. Referring to Eq. (4.19), we see that the only non-zero components for $\tilde{\sigma}_{ij}$ are $\tilde{\sigma}_{22}$ and $\tilde{\sigma}_{33}$. The numerical calculations of $\tilde{\sigma}_{ij}$ for (0/90) composites are included in Tables 6-9 while that for (θ/-θ) composites are given in Table 10

We see from Table 6 that $\tilde{\sigma}_{22}$ for the (0/15) composite in Table 6 is many times larger than $\tilde{\sigma}_{33}$ and $\tilde{\sigma}_{33}'$. In other words, the tensile stress at the interface is many times larger than the applied axial extensional stress. Notice however that $\tilde{\sigma}_{ij}$ is not the total stress. The total stress consists of $\tilde{\sigma}_{ij}$ and $\sigma_{ij}^{(1)}$ as well as the $k^* r^\delta$ and k(ln r) terms.

In Figs. 10 and 11 we present the logarithmic stress intensity factor k for all possible combinations of (θ/θ'). The contour lines for constant k values are given only in one quarter of the (θ, θ') plane since the contour lines in the remaining plane are a repetition of the ones shown in the figures. As we pointed out earlier in the analysis, Figs. 10 and 11 indicate that k = 0 occurs for (0/90) and (θ/-θ) composites. Of course, k is also zero for (θ/θ) composites in which the two materials are the same across the interface. However, we find an

additional contour line for $k = 0$ in Figs. 10 and 11 which lies slightly above the line $\theta' = 0$. This is unexpected and was not obvious from the analysis presented earlier.

In conclusion, it should be pointed out that the singular stress of the type $k^* r^\delta$ obtained in (2.27) should be added to (4.54) to obtain the singularities at the free edge. Using (4.58), we may write the stress singularity near the free edge in the form

$$\sigma_{ij} = k^* r^\delta \sigma_{ij}^* + [k\varepsilon_3(\ln r) + \alpha_1]\sigma_{ij}^{(1)} + \varepsilon_3 \tilde{\sigma}_{ij} \qquad (4.62)$$

where $\delta < 0$, k^*, k, α_1 and $\sigma_{ij}^{(1)}$ are constants while σ_{ij}^* and $\tilde{\sigma}_{ij}$ depend on ϕ. The analysis presented here provides every term on the right of (4.62) except k^* and α_1 which have to be determined by solving the complete boundary value problem.

Finally, it should also be pointed out that even though the layers are assumed to be of the same orthotropic material for the numerical calculations in this report, the theory presented here applies to any anisotropic layered composite.

TABLE 6

$\tilde{\sigma}_{ij}$ of Eq. (4.58) for Composite W, (0/θ')

θ'	15°	30°	45°	60°	75°	90°
k	0.0022	0.0033	0.0042	0.0040	0.0025	0
$\tilde{\sigma}_{11}(0)$	0.0221	0.0145	0.0102	0.0056	0.0016	0
$\tilde{\sigma}_{11}'(0)$	-0.0222	-0.0149	-0.0112	-0.0067	-0.0021	0
$\tilde{\sigma}_{22}(90)$	190.43	29.599	8.0106	1.2162	-1.3911	-2.1000
$\tilde{\sigma}_{22}(0)$	190.45	29.613	8.0208	1.2218	-1.3895	-2.1000
$\tilde{\sigma}_{22}'(-90)$	190.47	29.628	8.0310	1.2274	-1.3880	-2.1000
$\tilde{\sigma}_{33}(90)$	59.990	26.216	21.682	20.256	19.708	19.559
$\tilde{\sigma}_{33}(0)$	60.000	26.222	21.687	20.258	19.708	19.559
$\tilde{\sigma}_{33}'(0)$	35.132	7.1872	3.4312	2.2440	1.7848	1.6590
$\tilde{\sigma}_{33}'(-90)$	35.137	7.1903	3.4340	2.2461	1.7856	1.6590
$\tilde{\sigma}_{23}(90)$	0	0	0	0	0	0
$\tilde{\sigma}_{23}(0)$	0	0	0	0	0	0
$\tilde{\sigma}_{23}'(-90)$	0.0003	0.0006	0.0007	0.0004	0	0
$\tilde{\sigma}_{13}(0)$	-0.0035	-0.0052	-0.0065	-0.0063	-0.0040	0
$\tilde{\sigma}_{13}'(0)$	0.0034	0.0048	0.0059	0.0059	0.0039	0
$\tilde{\sigma}_{12}(0)$	0.0140	0.0092	0.0065	0.0035	0.0010	0

TABLE 7

$\tilde{\sigma}_{ij}$ of Eq. (4.58) for Composite W, (90/θ')

θ'	0°	15°	30°	45°	60°	75°
k	0	0.6874	0.5030	0.2954	0.1297	0.0239
$\tilde{\sigma}_{11}(0)$	0	-0.3032	-1.0048	-1.3511	-1.1657	-0.4910
$\tilde{\sigma}_{11}'(0)$	0	0.2058	0.7794	1.1633	1.0875	0.4823
$\tilde{\sigma}_{22}(90)$	-2.1000	-4.5417	-4.9455	-4.2661	-3.3602	-2.4156
$\tilde{\sigma}_{22}(0)$	-2.1000	-4.6419	-5.2778	-4.7129	-3.7456	-2.5788
$\tilde{\sigma}_{22}'(-90)$	-2.1000	-4.7473	-5.6151	-5.1610	-4.1303	-2.7409
$\tilde{\sigma}_{33}(90)$	1.6590	1.1463	1.0615	1.2041	1.3944	1.5925
$\tilde{\sigma}_{33}(0)$	1.6590	1.1185	0.9695	1.0805	1.2877	1.5476
$\tilde{\sigma}_{33}'(0)$	19.559	11.311	5.5014	2.9133	1.8095	1.5966
$\tilde{\sigma}_{33}'(-90)$	19.559	7.8189	2.9701	1.7004	1.3932	1.5244
$\tilde{\sigma}_{23}(90)$	0	0	0	0	0	0
$\tilde{\sigma}_{23}(0)$	0	0	0	0	0	0
$\tilde{\sigma}_{23}'(-90)$	0	0.2870	0.3950	0.2754	0.1242	0.0222
$\tilde{\sigma}_{13}(0)$	0	-1.0797	-0.7902	-0.4641	-0.2038	-0.3762
$\tilde{\sigma}_{13}'(0)$	0	1.6460	1.8195	1.3607	0.6910	0.1382
$\tilde{\sigma}_{12}(0)$	0	-0.0942	-0.3122	-0.4197	-0.3621	-0.1525

TABLE 8

$\tilde{\sigma}_{ij}$ of Eq. (4.58) for Composite T, (0/θ')

θ'	15°	30°	45°	60°	75°	90°
k	0.0349	0.0610	0.0729	0.0658	0.0394	0
$\tilde{\sigma}_{11}(0)$	0.2203	0.1919	0.1335	0.0695	0.0192	0
$\tilde{\sigma}_{11}'(0)$	-0.2218	-0.1990	-0.1467	-0.0817	-0.0238	0
$\tilde{\sigma}_{22}(90)$	62.232	13.729	3.3929	0.0314	-1.2148	-1.5400
$\tilde{\sigma}_{22}(0)$	67.452	13.921	3.5264	0.1009	-1.1956	-1.5400
$\tilde{\sigma}_{22}'(-90)$	67.672	14.115	3.6623	0.1729	-1.1753	-1.5400
$\tilde{\sigma}_{33}(90)$	40.825	25.844	22.950	22.009	21.660	21.569
$\tilde{\sigma}_{33}(0)$	40.948	25.952	23.025	22.048	21.671	21.569
$\tilde{\sigma}_{33}'(0)$	22.241	6.0686	2.8463	1.7151	1.2441	1.1088
$\tilde{\sigma}_{33}'(-90)$	22.260	6.0653	2.8494	1.7218	1.2474	1.1088
$\tilde{\sigma}_{23}(90)$	0	0	0	0	0	0
$\tilde{\sigma}_{23}(0)$	0	0	0	0	0	0
$\tilde{\sigma}_{23}'(-90)$	0.0036	0.0115	0.0118	0.0062	0.0011	0
$\tilde{\sigma}_{13}(0)$	-0.0549	-0.0959	-0.1145	-0.1034	-0.0619	0
$\tilde{\sigma}_{13}'(0)$	0.0498	0.0893	0.1076	0.0997	0.0612	0
$\tilde{\sigma}_{12}(0)$	0.1444	0.1258	0.0875	0.0455	0.0126	0

TABLE 9

$\tilde{\sigma}_{ij}$ of Eq. (4.58) for Composite T, (90/θ')

θ'	0°	15°	30°	45°	60°	75°
k	0	0.7217	0.5122	0.2993	0.1331	0.0252
$\tilde{\sigma}_{11}(0)$	0	-0.4729	-1.2109	-1.5256	-1.2991	-0.5560
$\tilde{\sigma}_{11}'(0)$	0	0.3070	0.9159	1.3004	1.2090	0.5460
$\tilde{\sigma}_{22}(90)$	-1.5400	-2.3081	-2.1663	-1.5927	-0.8749	-0.0430
$\tilde{\sigma}_{22}(0)$	-1.5400	-2.4380	-2.4991	-2.0120	-1.2320	-0.1958
$\tilde{\sigma}_{22}'(-90)$	-1.5400	-2.5715	-2.8316	-2.4273	-1.5858	-0.3481
$\tilde{\sigma}_{33}(90)$	1.1088	0.8937	0.9334	1.0940	1.2950	1.5280
$\tilde{\sigma}_{33}(0)$	1.1088	0.8481	0.8165	0.9467	1.1696	1.4743
$\tilde{\sigma}_{33}'(0)$	21.569	12.089	5.7896	3.0108	1.8057	1.5557
$\tilde{\sigma}_{33}'(-90)$	21.569	7.9469	2.8828	1.6049	1.3092	1.4674
$\tilde{\sigma}_{23}(90)$	0	0	0	0	0	0
$\tilde{\sigma}_{23}(0)$	0	0	0	0	0	0
$\tilde{\sigma}_{23}'(-90)$	0	0.3258	0.4190	0.2847	0.1286	0.0237
$\tilde{\sigma}_{13}(0)$	0	-1.1337	-0.8046	-0.4701	-0.2090	-0.0397
$\tilde{\sigma}_{13}'(0)$	0	1.7992	1.9660	1.4739	0.7590	0.1559
$\tilde{\sigma}_{12}(0)$	0	-0.1349	-0.3455	-0.4353	-0.3707	-0.1586

TABLE 10

$\tilde{\sigma}_{22}$ and $\tilde{\sigma}_{33}$ of Eq. (4.61) for (θ/-θ) Composites

	θ	15°	30°	45°	60°	75°
Composite	$\tilde{\sigma}_{22} = \tilde{\sigma}_{22}'$	191.14	29.887	8.2424	1.4187	-1.2079
W	$\tilde{\sigma}_{33} = \tilde{\sigma}_{33}'$	35.231	7.2214	3.4618	2.2766	1.8203
Composite	$\tilde{\sigma}_{22} = \tilde{\sigma}_{22}'$	71.464	16.660	5.9075	2.2697	0.8352
T	$\tilde{\sigma}_{33} = \tilde{\sigma}_{33}'$	23.032	6.5434	3.3082	2.2137	1.7821

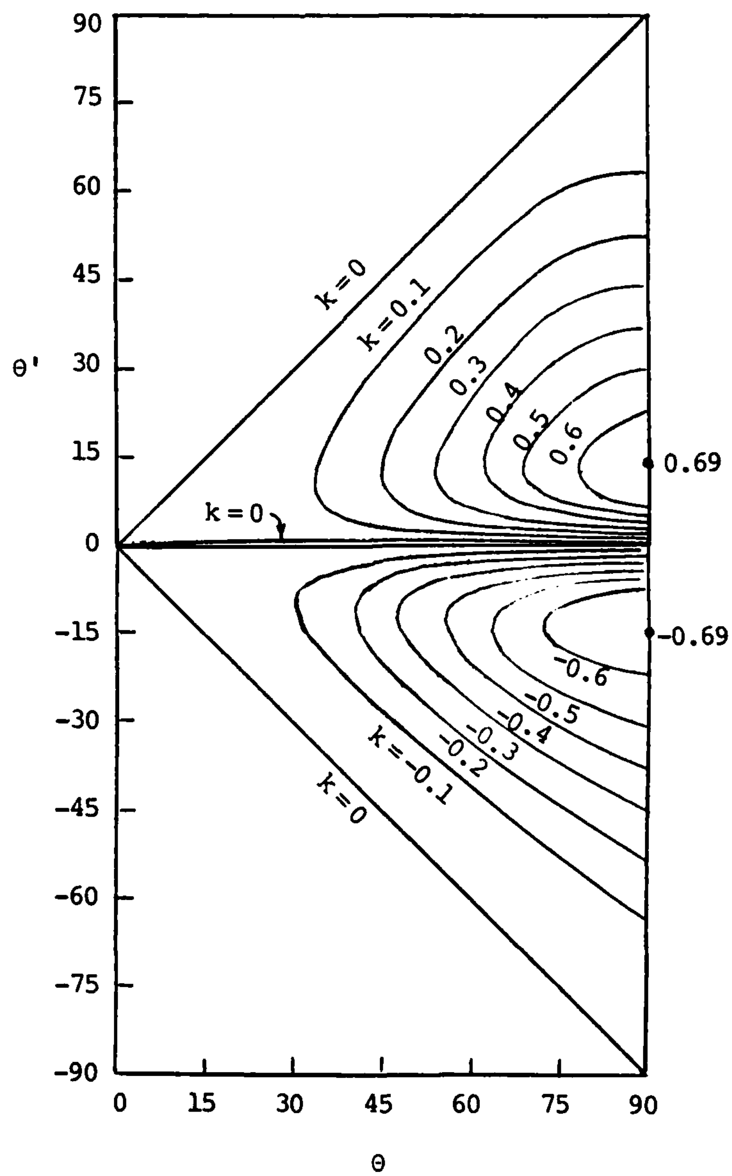

Fig. 10 k of the k(ln r) singularity at the free-edge for composite W

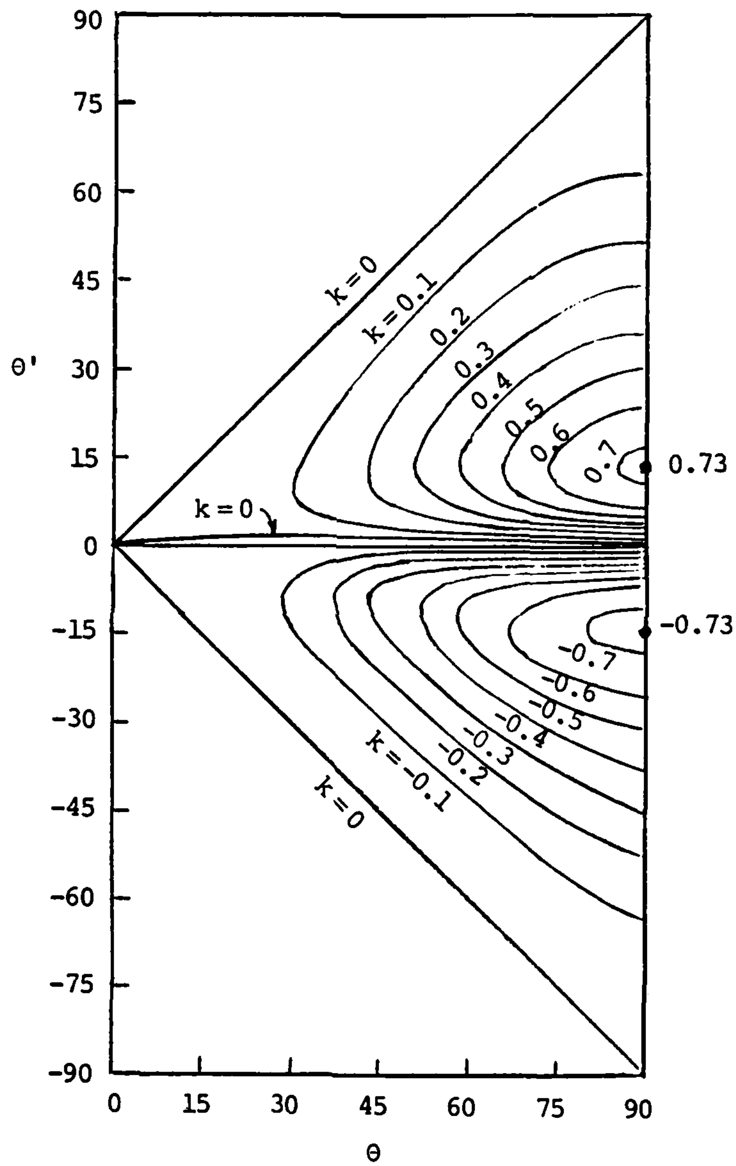

Fig. 11 k of the k(ln r) singularity at the free-edge for composite T

Chapter V

A CONTACT PROBLEM IN COMPOSITES

5.1 TWO-DIMENSIONAL PROBLEM

In Fig. 2, the edge $x_1 = 0$ of the composite was assumed stress free. If a rigid body is pressed against the face $x_1 = 0$, we would have the deformation illustrated in Fig. 12. Assuming that the rigid body is smooth and friction free, the boundary conditions on $x_1 = 0$ and near the origin are now given by

$$\left.\begin{aligned} &\sigma_{12} = 0, \qquad \sigma_{13} = 0 \\ &u_1 = U + wx_2 + \ldots \end{aligned}\right\} \tag{5.1}$$

where U and w are constants, and the dots stand for terms of order equal to or higher than $x_2{}^2$. The first two terms for u_1 can be included in a rigid body displacement as we discussed in Chapter III. Therefore, for the purpose of finding the singularity at the origin, we may replace Eq. (5.1) by

$$u_1 = \sigma_5 = \sigma_6 = 0 \tag{5.2}$$

The eigenvalues p_L and the associated eigenvectors υ_i, τ_{ij}, as well as the expressions for stress and displacement are the same as in Chapter II, Eqs. (2.4-2.7). By applying the interface continuity condition Eq. (2.13) at $\phi = 0$, and the new boundary conditions Eq. (5.2)

at $\phi = \pm\pi/2$ to Eqs. (2.10, 2.11), we obtain a new system of 12 linear homogeneous equations for A_L, B_L, A_L', B_L' which can be written as

$$\hat{\underset{\sim}{K}}_c \underset{\sim}{q} = \underset{\sim}{0} \tag{5.3}$$

where the hat is used to distinguish this new matrix from the previous $\underset{\sim}{K}_c$ matrix, and $\underset{\sim}{q}$ is again a column matrix whose elements are the complex constants A_L, B_L, A_L', B_L'. If δ is real then Eqs. (2.17, 2.18) may be used. The system of equations obtained by applying the boundary and interface continuity conditions to these equations is

$$\hat{\underset{\sim}{K}} \underset{\sim}{a} = \underset{\sim}{0} \tag{5.4}$$

where $\hat{\underset{\sim}{K}}$ is a real matrix, and $\underset{\sim}{a}$ is a column matrix whose elements are the real constants a_L, $\tilde{a}_L$, a_L', $\tilde{a}_L'$. For both systems, Eqs. (5.3, 5.4), a nontrivial solution exists if $\|\hat{\underset{\sim}{K}}_c\| = 0$. Equation (5.3) is solved by the method described in section 2.2 for complex δ, while a simpler and less time consuming method is used to find real δ from Eq. (5.4). Values found for δ by both methods are listed in Tables 11-14. The negative real values of δ, ($-1 < \delta < 0$) lead to a stress singularity at $r = 0$.

5.2 THREE-DIMENSIONAL PROBLEM

As we discussed in Chapter III, a three-dimensional deformation is, to the first order of approximation, a uniform extension ε_3 in the x_3-direction superimposed on the two-dimensional plane strain deformation. Similar to the free-edge problem considered in Chapter IV, we will first assume that a homogeneous stress solution exists due to a uniform extensional strain ε_3. The problem will be analyzed by two

methods. The first method enables us to see easily if a homogeneous stress solution exists. The second method enables us to extend the analysis, if necessary, to the case when a homogeneous stress solution is impossible.

5.3 HOMOGENEOUS STRESS SOLUTION: METHOD 1

The boundary condition $u_1 = 0$ on $x_1 = 0$ of Eq. (5.2) implies that

$$\partial u_1/\partial x_2 = \partial u_1'/\partial x_2 = 0 \tag{5.5}$$

along $x_1 = 0$. If the stress, and hence the strain, is homogeneous, Eq. (5.5) also applies along $x_2 = 0$. Now

$$\varepsilon_6 = (\partial u_2/\partial x_1)/2, \qquad \varepsilon_6' = (\partial u_2'/\partial x_1)/2 \tag{5.6}$$

and since u_2 is continuous along $x_2 = 0$,

$$[\varepsilon_6] = 0 \tag{5.7}$$

For a homogeneous stress solution therefore, Eq. (5.7) replaces the condition $u_1 = 0$ along $x_1 = 0$. Application of the material symmetry properties of Eq. (4.4) and the boundary conditions of Eq. (5.2) to Eq. (1.3) for ε_1, ε_3, ε_5, and ε_6 results in

$$\varepsilon_1 = s_{11}\sigma_1 + s_{12}\sigma_2 + s_{13}\sigma_3 \tag{5.8}$$

$$\varepsilon_3 = s_{31}\sigma_1 + s_{32}\sigma_2 + s_{33}\sigma_3 \tag{5.9}$$

$$\varepsilon_5 = s_{51}\sigma_1 + s_{52}\sigma_2 + s_{53}\sigma_3 \tag{5.10}$$

$$\varepsilon_6 = s_{64}\sigma_4 \tag{5.11}$$

Solving for σ_3 from Eq. (5.9), and eliminating σ_3 from the remaining equations, Eqs. (5.8, 5.10) we have

$$\sigma_3 = \varepsilon_3/s_{33} - (s_{31}/s_{33})\sigma_1 - (s_{32}/s_{33})\sigma_2 \tag{5.12}$$

$$\varepsilon_1 = (s_{11} - s_{31}s_{13}/s_{33})\sigma_1 + (s_{12} - s_{32}s_{13}/s_{33})\sigma_2 + (s_{13}/s_{33})\varepsilon_3 \tag{5.13}$$

$$\varepsilon_5 = (s_{51} - s_{31}s_{53}/s_{33})\sigma_1 + (s_{52} - s_{32}s_{53}/s_{33})\sigma_2 + (s_{53}/s_{33})\varepsilon_3 \tag{5.14}$$

Let

$$\left.\begin{aligned} R_1 &= s_{13}/s_{33} \\ R_5 &= s_{53}/s_{33} \\ R_{11} &= s_{11} - s_{31}s_{13}/s_{33} \\ R_{12} &= s_{12} - s_{13}s_{32}/s_{33} \\ R_{51} &= s_{51} - s_{53}s_{31}/s_{33} \\ R_{52} &= s_{52} - s_{53}s_{32}/s_{33} \end{aligned}\right\} \tag{5.15}$$

If we apply the interface conditions Eqs.(5.7) and (4.14) to Eqs. (5.11, 5.13, 5.14), we obtain

$$[s_{64}]\sigma_4 = 0 \tag{5.16}$$

$$R_{11}\sigma_1 - R_{11}'\sigma_1' + [R_{12}]\sigma_2 = -[R_1]\varepsilon_3 \tag{5.17}$$

$$R_{51}\sigma_1 - R_{51}'\sigma_1' + [R_{52}]\sigma_2 = -[R_5]\varepsilon_3 \tag{5.18}$$

Equation (5.16) indicates that if $[s_{64}] = 0$, which is the case for (0/90) composites, σ_4 is arbitrary. Otherwise $\sigma_4 = 0$. For the composites considered here, $s_{64} = s_{64}' = 0$ for any angle ply. Hence σ_4 is arbitrary here, and will be denoted by the arbitrary constant α_1.

Another arbitrary solution is obtained from solving Eqs. (5.17, 5.18). If we let $\sigma_2 = \alpha_2$, we may solve for σ_1 and σ_1', while σ_3 and σ_3' are determined from Eq. (5.12). The result can be written in the following form:

$$\underset{\sim}{\sigma} = \varepsilon_3\underset{\sim}{\sigma}^{(p)} + \alpha_1\underset{\sim}{\sigma}^{(1)} + \alpha_2\underset{\sim}{\sigma}^{(2)} \tag{5.19}$$

where

$$\underset{\sim}{\sigma}^{(p)} = \begin{bmatrix} \sigma_1^{(p)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_3^{(p)} \end{bmatrix} \qquad \underset{\sim}{\sigma}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\underset{\sim}{\sigma}^{(2)} = \begin{bmatrix} \sigma_1^{(2)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma_3^{(2)} \end{bmatrix} \tag{5.20}$$

and

$$\left.\begin{aligned} \sigma_1^{(p)} &= -([R_1]R_{51}' - [R_5]R_{11}')/(R_{11}R_{51}' - R_{51}R_{11}') \\ \sigma_1^{(2)} &= -([R_{12}]R_{51}' - [R_{52}]R_{11}')/(R_{11}R_{51}' - R_{51}R_{11}') \\ \sigma_3^{(p)} &= -(1 - s_{31}\sigma_1^{(p)})/s_{33} \\ \sigma_3^{(2)} &= -(s_{31}\sigma_1^{(2)} + s_{32})/s_{33} \end{aligned}\right\} \tag{5.21}$$

Similar expressions can be obtained for $\underset{\sim}{\sigma}'$.

For $(\theta/-\theta)$ composites, $R_1 = R_1'$, $R_{11} = R_{11}'$, $R_{12} = R_{12}'$, $R_5 = -R_5'$, $R_{51} = -R_{51}'$ and $R_{52} = -R_{52}'$. $\sigma_1^{(p)}$ and $\sigma_1^{(2)}$ in Eq. (5.21) are reduced to

$$\sigma_1^{(p)} = -R_5/R_{51}, \qquad \sigma_1^{(2)} = -R_{52}/R_{51} \tag{5.22}$$

Moreover, $\underset{\sim}{\sigma}^{(p)} = \underset{\sim}{\sigma}'^{(p)}$, $\underset{\sim}{\sigma}^{(2)} = \underset{\sim}{\sigma}'^{(2)}$ and hence $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}'$. In Tables 15 and 16 we list $\sigma_1^{(p)}$, $\sigma_3^{(p)}$, $\sigma_1^{(2)}$, and $\sigma_3^{(2)}$ for both composites of $(\theta/-\theta)$ ply.

For $(0/\theta')$ composites in Tables 17 and 18, $\sigma_1^{(p)} = \sigma_1'^{(p)}$ while $\sigma_3^{(p)}$ and $\sigma_3'^{(p)}$ are quite different. Tables 19 and 20 show the results for $(90/\theta')$ composites; in this case $\sigma_3^{(2)}$ is nearly equal to $\sigma_3'^{(2)}$, while $\sigma_1^{(2)}$ and $\sigma_1'^{(2)}$ are quite different. Both $\sigma_1^{(p)}$ and $\sigma_3^{(p)}$ in Tables 15-20 have the unit of 10^6 psi.

For (0/90) composites, $R_{51} = R_{51}' = 0$ and $\sigma_1^{(2)}$, $\sigma_1^{(p)}$ of Eq. (5.21) do not exist. In fact $R_{52} = R_{52}' = R_5 = R_5' = 0$ also, and Eq. (5.18) is trivially satisfied. Equation (5.16) and (5.17) then yield three arbitrary solutions. Letting $\sigma_4 = \alpha_1$, $\sigma_2 = \alpha_2$ and $\sigma_1' = \alpha_3$, the solution can be written as

$$\underset{\sim}{\sigma} = \varepsilon_3 \underset{\sim}{\sigma}^{(p)} + \alpha_1 \underset{\sim}{\sigma}^{(1)} + \alpha_2 \underset{\sim}{\sigma}^{(2)} + \alpha_3 \underset{\sim}{\sigma}^{(3)} \tag{5.23}$$

where $\underset{\sim}{\sigma}^{(p)}$, $\underset{\sim}{\sigma}^{(1)}$, and $\underset{\sim}{\sigma}^{(2)}$ are identical to the ones defined in Eqs. (5.20, 5.21) except $\sigma_1^{(p)}$ and $\sigma_1^{(2)}$ which are replaced by

$$\sigma_1^{(p)} = -[R_1]/R_{11}, \qquad \sigma_1^{(2)} = -[R_{12}]/R_{11} \tag{5.24}$$

and $\underset{\sim}{\sigma}^{(3)}$ is

$$\underset{\sim}{\sigma}^{(3)} = \begin{bmatrix} \sigma_1^{(3)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_3^{(3)} \end{bmatrix} \tag{5.25}$$

where

$$\sigma_1^{(3)} = R_{11}'/R_{11}, \qquad \sigma_3^{(3)} = -\sigma_1^{(3)} s_{31}/s_{33} \tag{5.26}$$

For $\underset{\sim}{\sigma}'$, the components of $\underset{\sim}{\sigma}'^{(p)}$, $\underset{\sim}{\sigma}'^{(2)}$ and $\underset{\sim}{\sigma}'^{(3)}$ are given by

$$\begin{gathered}\sigma_1'^{(p)} = 0, \quad \sigma_1'^{(2)} = 0, \quad \sigma_1'^{(3)} = 1 \\ \sigma_3'^{(p)} = 1/s_{33}', \quad \sigma_3'^{(2)} = -s_{32}'/s_{33}', \quad \sigma_3'^{(3)} = -s_{31}'/s_{33}'\end{gathered} \tag{5.27}$$

In Table 21 we list the result for (0/90) composites. For the composites considered here $\underset{\sim}{\sigma}^{(3)} = \underset{\sim}{\sigma}'^{(3)}$.

It appears that Eqs. (5.17, 5.18) always have a solution unless one of the following relations is satisfied:

$$0 = R_{11} = R_{11}' = [R_{12}] \neq [R_1] \tag{5.28}$$

$$0 = R_{51} = R_{51}' = [R_{52}] \neq [R_5] \tag{5.29}$$

$$R_{11}/R_{51} = R_{11}'/R_{51}' = [R_{12}]/[R_{52}] \neq [R_1]/[R_5] \tag{5.30}$$

If anyone or more of the above three equations is satisfied, Eq. (5.17) and/or (5.18) are inconsistent and a solultion does not exist. For the composites considered here, none of Eqs. (5.28, 5.29, 5.30) are satisfied for all possible combinations of (θ/θ') angles. Therefore, a homogeneous stress solution always exists for a uniform extension ε_3.

5.4 HOMOGENEOUS STRESS SOLUTION: METHOD 2

Equations for stress and displacement are the same as Eqs. (4.28, 4.29). Upon applying the boundary conditions of Eq. (5.2) at $\phi = \pm\pi/2$, and the interface continuity conditions of Eq. (2.13) at $\phi = 0$, we obtain the system of equations

$$\hat{\underset{\sim}{K}}\underset{\sim}{a} = \underset{\sim}{b}\varepsilon_3 \tag{5.31}$$

which have a non-trivial solution if $\underset{\sim}{\ell}^T \underset{\sim}{b} = 0$, where $\underset{\sim}{\ell}^T \hat{\underset{\sim}{K}} = \underset{\sim}{0}$. For the composites considered here $\underset{\sim}{\ell}^T \underset{\sim}{b} = 0$ for all (θ/θ') ply, and the numerical solutions agree with those obtained by method 1.

TABLE 11

Negative Real Roots δ for r^{δ} Terms at the Contact-Edge in Composite W

θ	δ		
	$\theta' = 0$	$\theta' = 90°$	$\theta' = -\theta$
15°	-9.3671×10^{-4}	-4.6536×10^{-4}	-4.6679×10^{-3}
30°	-1.2238×10^{-2}	-6.0849×10^{-3}	-6.8300×10^{-2}
45°	-3.6434×10^{-2}	-1.8759×10^{-2}	-1.7190×10^{-1}
60°	-5.5918×10^{-2}	-3.0693×10^{-2}	-2.0388×10^{-1}
75°	-3.8771×10^{-2}	-2.3686×10^{-2}	-1.1389×10^{-1}

TABLE 12

Negative Real Roots δ for r^{δ} Terms at the Contact-Edge in Composite T

θ	δ		
	$\theta' = 0$	$\theta' = 90°$	$\theta' = -\theta$
15°	-2.5704×10^{-3}	-1.1110×10^{-3}	-1.2894×10^{-2}
30°	-1.9977×10^{-2}	-8.6484×10^{-3}	-1.0695×10^{-1}
45°	-5.0588×10^{-2}	-2.3031×10^{-2}	-2.2038×10^{-1}
60°	-7.1066×10^{-2}	-3.5292×10^{-2}	-2.3995×10^{-1}
75°	-4.6735×10^{-2}	-2.6708×10^{-2}	-1.3011×10^{-1}

TABLE 13

Complex Roots* δ for r^δ Terms at the Contact-Edge in Composite W

θ	δ_1	δ_2	δ_3	δ_4
15/-15	-.005	.995 + .100 i	1.996 + .120 i	2.994 + .180 i
0/15	-.001	.999 + .055 i	2.0 + .053 i	2.998 + .099 i
90/15	-.001	.998 + .065 i	1.998 + .049 i	
60/-60	-.204	.701 + .721 i	1.677 + .882 i	2.449 + 1.453 i
0/60	-.056	.896 + .413 i	2.007 + .359 i	2.567 + .919 i
90/60	-.037	.840 + .524 i	1.985 + .271 i	2.657 + 1.133 i

TABLE 14

Complex Roots* δ for r^δ Terms at the Contact-Edge in Composite T

θ	δ_1	δ_2	δ_3	δ_4
15/-15	-.013	.989 + .156 i	1.994 + .185 i	2.993 + .268 i
0/15	-.003	.997 + .081 i	1.999 + .081 i	2.997 + .150 i
90/15	-.001	.996 + .103 i	1.997 + .078 i	
60/-60	-.240	.649 + .771 i	1.621 + .948 i	2.359 + 1.558 i
0/60	-.071	.887 + .436 i	2.030 + .363 i	
90/60	-.035	.803 + .575 i	1.987 + .274 i	2.572 + 1.246 i

* zero and positive integers are also roots for δ

TABLE 15

$\sigma_1^{(p)}$, $\sigma_3^{(p)}$, $\sigma_1^{(2)}$, $\sigma_3^{(2)}$ of Eqs. (5.21, 5.22) for Composite W, (θ/-θ)

θ	15°	30°	45°	60°	75.°
$\sigma_1^{(p)} = \sigma_1'^{(p)}$	-28.239	-5.8883	-1.7000	-0.2970	0.2543
$\sigma_3^{(p)} = \sigma_3'^{(p)}$	-3.4539	0.8866	1.7000	1.9725	2.0795
$\sigma_1^{(2)} = \sigma_1'^{(2)}$	0.1477	0.1970	0.2063	0.2093	0.2106
$\sigma_3^{(2)} = \sigma_3'^{(2)}$	0.2024	0.2120	0.2137	0.2143	0.2146

TABLE 16

$\sigma_1^{(p)}$, $\sigma_3^{(p)}$, $\sigma_1^{(2)}$, $\sigma_3^{(2)}$ of Eqs. (5.21, 5.22) for Composite T, (θ/-θ)

θ	15°	30°	45°	60°	75°
$\sigma_1^{(p)} = \sigma_1'^{(p)}$	-15.948	-4.4035	-1.6200	-0.6304	-0.2332
$\sigma_3^{(p)} = \sigma_3'^{(p)}$	2.4089	1.7733	1.6200	1.5655	1.5436
$\sigma_1^{(2)} = \sigma_1'^{(2)}$	0.2232	0.2643	0.2742	0.2778	0.2792
$\sigma_3^{(2)} = \sigma_3'^{(2)}$	0.2886	0.2863	0.2858	0.2792	0.2855

TABLE 17

$\sigma_1^{(p)}$, $\sigma_3^{(p)}$, $\sigma_1^{(2)}$, $\sigma_3^{(2)}$ of Eq. (5.21) for Composite W, (0/θ')

θ'	15°	30°	45°	60°	75°
$\sigma_1^{(p)}$	-28.206	-5.8551	-1.6667	-0.2637	0.2877
$\sigma_1'^{(p)}$	-28.239	-5.8883	-1.7000	-0.2970	0.2543
$\sigma_3^{(p)}$	14.077	18.770	19.650	19.945	20.060
$\sigma_3'^{(p)}$	-3.4539	0.8866	1.7000	1.9725	2.0795
$\sigma_1^{(2)}$	0.1478	0.1971	0.2063	0.2094	0.2106
$\sigma_1'^{(2)}$	0.1477	0.1970	0.2063	0.2093	0.2106
$\sigma_3^{(2)}$	0.2410	0.2514	0.2533	0.2540	0.2542
$\sigma_3'^{(2)}$	0.2024	0.2120	0.2137	0.2143	0.2146

TABLE 18

$\sigma_1^{(p)}$, $\sigma_3^{(p)}$, $\sigma_1^{(2)}$, $\sigma_3^{(2)}$ of Eq. (5.21) for Composite T, (0/θ')

θ'	15°	30°	45°	60°	75°
$\sigma_1^{(p)}$	-15.565	-3.9220	-1.1149	-0.1169	0.2837
$\sigma_1'^{(p)}$	-15.948	-4.4035	-1.6200	-0.6304	-0.2332
$\sigma_3^{(p)}$	17.642	20.902	21.688	21.967	22.079
$\sigma_3'^{(p)}$	2.4089	1.7733	1.6200	1.5655	1.5436
$\sigma_1^{(2)}$	0.2245	0.2660	0.2760	0.2796	0.2810
$\sigma_1'^{(2)}$	0.2232	0.2643	0.2742	0.2778	0.2792
$\sigma_3^{(2)}$	0.3429	0.3545	0.3573	0.3583	0.3587
$\sigma_3'^{(2)}$	0.2886	0.2863	0.2858	0.2856	0.2855

TABLE 19

$\sigma_1^{(p)}$, $\sigma_3^{(p)}$, $\sigma_1^{(2)}$, $\sigma_3^{(2)}$ of Eq. (5.21) for Composite W, (90/θ')

θ'	15°	30°	45°	60°	75°
$\sigma_1^{(p)}$	-272.41	-59.540	-19.650	-6.2877	-1.0366
$\sigma_1'^{(p)}$	-28.239	-5.8883	-1.7000	-0.2970	0.2543
$\sigma_3^{(p)}$	-3.9066	0.7872	1.6667	1.9614	2.0771
$\sigma_3'^{(p)}$	-3.4539	0.8866	1.7000	1.9724	2.0795
$\sigma_1^{(2)}$	-.03907	0.0787	0.1667	0.1961	0.2077
$\sigma_1'^{(2)}$	0.1477	0.1970	0.2063	0.2093	0.2106
$\sigma_3^{(2)}$	0.2014	0.2117	0.2137	0.2143	0.2146
$\sigma_3'^{(2)}$	0.2024	0.2119	0.2137	0.2143	0.2146

TABLE 20

$\sigma_1^{(p)}$, $\sigma_3^{(p)}$, $\sigma_1^{(2)}$, $\sigma_3^{(2)}$ of Eq. (5.21) for Composite T, (90/θ')

θ'	15°	30°	45°	60°	75°
$\sigma_1^{(p)}$	-228.12	-61.789	-21.688	-7.4310	-1.7076
$\sigma_1^{\prime(p)}$	-15.948	-4.4035	-1.6200	-0.6304	-0.2332
$\sigma_3^{(p)}$	-2.9311	0.3289	1.1149	1.3943	1.5065
$\sigma_3^{\prime(p)}$	2.4089	1.7733	1.6200	1.5656	1.5436
$\sigma_1^{(2)}$	-0.5329	0.0598	0.2027	0.2535	0.2739
$\sigma_1^{\prime(2)}$	0.2232	0.2643	0.2742	0.2778	0.2791
$\sigma_3^{(2)}$	0.2695	0.2812	0.2840	0.2850	0.2854
$\sigma_3^{\prime(2)}$	0.2886	0.2863	0.2860	0.2856	0.2855

TABLE 21

$\sigma_1^{(p)}$, $\sigma_3^{(p)}$, $\sigma_1^{(2)}$, $\sigma_3^{(2)}$, $\sigma_1^{(3)}$, $\sigma_3^{(3)}$
of Eqs. (5.21, 5.24, 5.26) for Composites, (0/90)

(0/90)	Composite 1	Composite 2
$\sigma_1^{(p)} \times 10^6$psi	0.3965	0.4032
$\sigma_1'^{(p)} \times 10^6$psi	0	0
$\sigma_3^{(p)} \times 10^6$psi	20.083	22.113
$\sigma_3'^{(p)} \times 10^6$psi	2.1000	1.5400
$\sigma_1^{(2)}$	0.1888	0.2618
$\sigma_1'^{(2)}$	0	0
$\sigma_3^{(2)}$	0.2497	0.3533
$\sigma_3'^{(2)}$	0.2100	0.2800
$\sigma_1^{(3)}$	0.1050	0.0700
$\sigma_1'^{(3)}$	1	1
$\sigma_3^{(3)} = \sigma_3'^{(3)}$	0.0220	0.0196

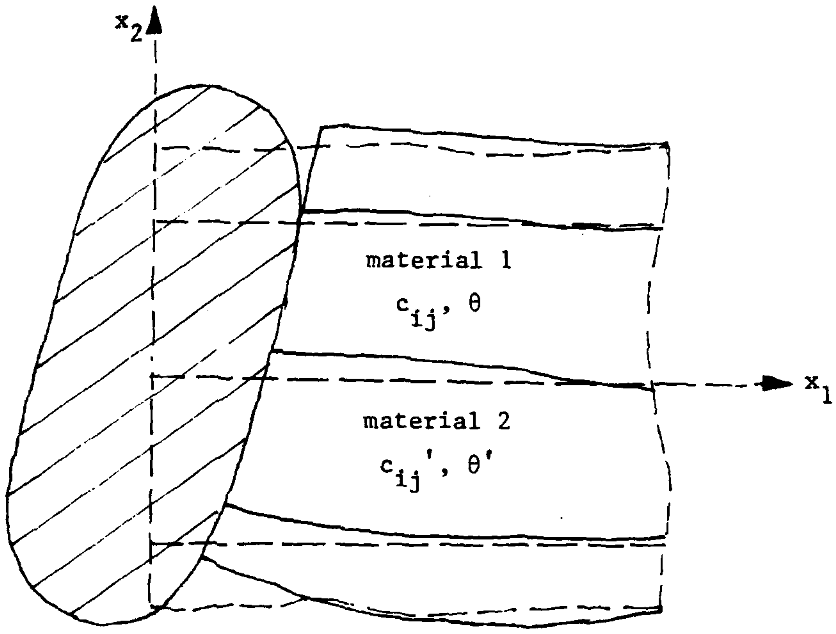

Fig. 12

A contact-edge between two adjacent layers (θ/θ')

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FREE-EDGE AND CONTACT-EDGE SINGULARITIES
IN LAMINATED COMPOSITES
R. I. Zwiers and T. C. T. Ting
Department of Materials Engineering
University of Illinois at Chicago Circle
Chicago, Illinois 60680
Technical Report AMMRC TR 82-21, April 1982, 84 pp
illus.-tables, Contract DAAG46-80-C-0081
D/A Project: 8X363304D215;
AMCMS Code: 633304.21503
Final Report, August 1980 to August 1981

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Key Words

Composite materials
Stress concentrations
Contact stress
Complex variables
Eigenvalues
Anisotropy
Elastic Analysis

This effort is part of the predictive response capability development to provide design criteria for space vehicles. For example, mechanical joints in structure have stress risers around the bolt hole due to the singularities from linear analysis. In this task the stress singularities at the free-edge and at the contact-edge of an interface in a laminated composite are studied. The composite is subjected to a uniform extension in the direction parallel to the plane of laminate layers. Two types of stess singularities are found. For the free-edge problem the stress solution for some fiber orientations contains only a singularity of k^*r^δ, while the solution for other combinations of fiber orientation contains an additional singularity of $k(\ln r)$. For the contact-edge problem only k^*r^δ singularities occur for all fiber orientations.

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